COMPLEX HELE-SHAW MODEL.
LOCAL SOLVABILITY FOR A DOUBLY CONNECTED DOMAIN

SERGEI V. ROGOSIN AND TATYANA S. VAITEKHOVICH

Department of Mathematics and Mechanics, Belarusian State University
F. Skaryny av. 4, 220050 Minsk, Belarus
e-mail: rogosin@bsu.by

Abstract. The paper is devoted to the study of the Hele-Shaw moving boundary value problem in doubly-connected domains. It is reformulated as a couple of problems, namely, the Schwarz boundary value problem for a doubly-connected domain and an abstract Cauchy-Kovalevsky problem. Local existence and uniqueness of the classical solution to the Hele-Shaw problem is shown by using a variant of the Schwarz alternation method for the Schwarz problem and Nirenberg-Nishida theorem for the Cauchy-Kovalevsky problem.

1. Introduction

Hele-Shaw model as discussed in the classical paper by british physicist H.S.Hele-Shaw [15] consists in the description of the dynamics of the frontier between two immiscible fluids in a narrow channel. In [38], [31] a mathematical model was derived for describing Hele-Shaw flows with a free boundary produced by injection/suction of fluid into/from a narrow channel supposing constant atmospheric pressure on the moving boundary. This model can be represented in the following form, where \( f = f(z, t) \) is the Riemann mapping function from the unit disk \( G_1 \) onto the region occupied by fluid at the time \( t \), and \( Q(t) \) is the rate of injection/suction:

\[
\text{Re} \left( \frac{1}{z} \frac{\partial f}{\partial t} \right) = Q(t), \quad z \in \partial G_1, \quad t \in [0, T),
\]

(1) \label{model1}

\[
f(z, 0) = f_0(z), \quad z \in G_1,
\]

(2) \label{model2}

\[
f(0, t) = 0, \quad t \in [0, T).
\]

(3) \label{model3}

This model commonly called complex Hele-Shaw model has been generalized in a number of direction (see e.g. [16], [36]). Local existence and uniqueness for (1)-(3) is shown in [38] under additional condition \( f_z'(0, t) > 0 \). In the case of rational initial mapping \( f_0(z) \) an elementary proof of the local existence is proposed in [12]. In [30] the complex Hele-Shaw model is reformulated as an abstract Cauchy-Kovalevsky problem. The solution of the later was found on the the base of the Nirenberg-Nishida theorem [20], [21] (c.f. [23]). This approach is also used in [24], [28], [29] for analytical and numerical treatment of certain variants of the complex Hele-Shaw...
model. Among the results dealing with analytic solutions to the Hele-Shaw problem we have to mention the classical paper [25] in which a first collection of the exact solutions was constructed (c.f. also [17]).

The Hele-Shaw moving boundary value problem and its generalizations is studied on the base of certain weak or variational formulations starting from the results in [7], [13], exploiting the celebrated Baiocchi transformation [3]. These formulations of the Hele-Shaw model may be viewed also as an enthalpy formulation of the Stefan problem [8].

Several attempts was made to attack the Hele-Shaw problem in the case of when the fluid occupies a multiply-connected region. In [14] an ill-posed moving boundary problem for doubly connected domains in $\mathbb{R}^N$. It is proposed a weak formulation of this problem, the local in time solvability is shown. This problem is connected with Hele-Shaw model when $N = 2$ and with electro-chemical machining model as $N \leq 3$. Another model which also lead naturally to the moving boundary value problem in a multiply-connected domain is that of freezing/solidification model (see e.g. [2]). In [32], [33] the Hele-Shaw problem for different type of multiply-connected domains is reduced by applying the Cauchy transform to the functional equation in complex domains (for developing of this technique and other application of the method of the functional equations see [19]).

The idea of the constructing a conformal mapping from a canonical $N$-connected region to the fluid domain domain, especially automorphic functions invariant with respect to corresponding Schottky groups, is employed in [34] for the study of the singularity-driven Hele-Shaw flows of a multiply-connected fluid region.

In [5] the theory of algebraic curves and quadrature domains is used to construct exact solutions to the problem of the squeeze flow of multiply-connected fluid domains in a Hele-Shaw cell.

In our paper we consider a generalization of the problem (1)-(3) to the case of doubly connected domains: given a function $f_0(z)$, holomorphic and univalent in a neighbourhood of the domain

$$G(0) \equiv \{ z \in \mathbb{C} : |z - a_j| > r_j(0), j = 1, 2; |a_1 - a_2| > r_1(0) + r_2(0) \},$$

functions $Q_1(t), Q_2(t)$, continuous in a right-sided neighbourhood of $t = 0$, and a number $a \in [0, 2\pi)$, find $f(z, t)$, holomorphic and univalent in $z$ in a neighbourhood of the domain

$$G(t) \equiv \{ z \in \mathbb{C} : |z - a_j| > r_j(t), j = 1, 2; |a_1 - a_2| > r_1(t) + r_2(t) \},$$

$$\Gamma_j(t) \equiv \{ z \in \mathbb{C} : |z - a_j| = r_j(t) \}; j = 1, 2,$$

and continuously differentiable in a right-sided neighbourhood of $t = 0$, satisfying the following conditions:

$$\text{Re} \left( \frac{1}{z} \frac{\partial f}{\partial t} \right) = Q_j(t), \quad z \in \Gamma_j(t), j = 1, 2, \quad t \in [0, T),$$

$$f(z, 0) = f_0(z), \quad z \in G(0),$$
We follow here the strategy of the paper [29] in which an ill-posed Hele-Shaw problem for simply connected domain was treated by reduction to an abstract Cauchy-Kovalevsky problem and nonlinear Riemann-Hilbert-Poincare problem (see also [30], [35]). The case of doubly connected domains leads to certain additional difficulties we have to overcome.

First, it is known (see [11]) that the conformal mapping of a doubly connected domain onto canonical domain (in particular onto the exterior of two discs) depends on so called conformal modulus (in our case - the fraction of the radii of the circles $\Gamma_1(t), \Gamma_2(t)$). In our study we use an analytic dependence of the conformal modulus on small perturbation of $\Gamma_1(t), \Gamma_2(t)$ (see [18]).

Second, to get the unique conformal mapping of a doubly connected domain onto the canonical domain one have to add a (real) uniqueness condition which need to have a physical sense. It is proposed in the form (6).

Third, the problem (4)-(6) can be reduced to a couple of problems, namely, the Schwarz problem for a doubly connected domain $G(t)$ and evolution equation to be solved successively. The first of these problems has a “global” and stationary character. The later means that it should be solved for each fixed time instance, but the former means that we have to construct solution in the whole domain. Moreover we need to show the possibility to continue the obtained solution analytically into a neighbourhood of the initial domain and regularity dependence of the solution on the small perturbation of the boundary curves. These properties and the solution itself are obtained on the base of a variant of the Schwarz alternation method as proposed in [19] and by using the analyticity of the solution to the Schwarz problem with respect to the boundary curves shown in [26]. We have to remark that in the contrary to the simply connected case there no suitable exact formula for the solution to the Schwarz problem. The second problem, namely, evolution equation, in the contrary, has a “local” character, i.e. is studied in a neighbourhood of each boundary curve. This problem is reduced to an abstract problem which is solved by using the following theorem which is a tool for the study abstract nonlinear operator equations in a scale of Banach spaces

**Theorem 1.** ([21]) Let $\{\mathcal{B}_s, \| \cdot \|_s\}_{0 \leq s \leq 1}$ be a scale of Banach spaces, i.e. a family of continuously embedded Banach spaces such that for all $0 < s' \leq s \leq 1$ the norm of the canonical embedding $\mathbf{I}_{s \rightarrow s'} : \mathcal{B}_s \rightarrow \mathcal{B}_{s'}$ is not greater than 1 ($\| \mathbf{I}_{s \rightarrow s'} \| \leq 1$).

Let us consider the abstract Cauchy-Kovalevsky problem

$$\frac{dw}{dt} = \mathbf{L}(t, w), \quad w(0) = 0,$$

satisfying the following conditions in $\{\mathcal{B}_s, \| \cdot \|_s\}_{0 \leq s \leq 1}$ (where $C, K, R,$ and $T$ be positive constants independent of $s, s', t$):

(i) the nonlinear operator $\mathbf{L}(t, w)$ is continuous in $t$ and maps $[0, T] \times \{w \in \mathcal{B}_s : \|w\|_s \leq R\}$ into $\mathcal{B}_{s'}$ for all $0 < s' \leq s \leq 1$;
(ii) the continuous function \( L(t, 0) \) satisfies
\[
\|L(t, 0)\| \leq \frac{K}{1-s}, \text{ for all } 0 < s < 1;
\] (9)  

(iii) for all \( 0 < s' \leq s \leq 1 \), \( t \in [0, T] \), and \( w_1, w_2 \) belonging to the open ball \( \{w \in B_s : \|w\|_s < R\} \) we have
\[
\|L(t, w_1) - L(t, w_2)\|_{s'} \leq \frac{C}{s - s'} \|w_1 - w_2\|_s.
\] (10)  

Under these assumptions there exists one and only one solution
\[
w \in C^1([0, a_0(1-s)); B_s)_{0 < s < 1}, \|w\|_s < R,
\]
where \( a_0 \) is a suitable positive constant.

Our paper is organized as follows. We start (Sec. 2) with reformulation of the initial problem (4)-(6) in the equivalent form, namely, in the form of the linear Schwarz boundary value problem for a doubly connected domain and an abstract Cauchy-Kovalevsky problem to be solved successively. We also discuss the structure of the obtained couple of problems. In Sec. 3 a suitable scale of Banach spaces is constructed and its properties are studied. Solution of the Schwarz problem in a doubly connected domain is presented in Sec. 4. We conclude our investigation (Sec. 5) by the application of Theorem 1 to the Cauchy-Kovalevsky problem in the Banach scale defined in Sec. 3.

2. THE STRUCTURE OF THE PROBLEM

Let us consider the structure of problem (4)-(6). The equation (4) can be rewritten in the form
\[
\text{Re} \left( \frac{1}{z} \frac{\partial f}{\partial t}(z, t) \left( \frac{\partial f}{\partial z}(z, t) \right)^{-1} \right) = \left| \frac{\partial f}{\partial z}(z, t) \right|^{-2} Q_j(t), \quad z \in \Gamma_j(t), j = 1, 2,
\] (11)  

which have to be solved for each fixed \( t \in [0, T] \). It is known [1] that under certain additional condition (e.g. under condition (6) or the condition
\[
\text{Im} \left( \frac{1}{z} \frac{\partial f}{\partial t}(z, t) \left( \frac{\partial f}{\partial z}(z, t) \right)^{-1} \right) \bigg|_{z = z_0} = 0,
\]
where \( z_0 \) is a point in \( G(t) \) the problem (11) has a unique solution which is defined by means of the Schwarz operator for the circular doubly connected domain \( T \). Only few exact formulas are known for the Schwarz operator, namely the celebrated Villat’s formula for concentric annulus (see [1]) and a formula for multiply connected circular domain (a form of this formula is presented in [19]). The first form of the Schwarz operator is an integral operator with the Weierstrass zeta-function in the kernel, but in the last one the kernel is represented as a series in all elements of certain Schottky group. Both forms are not suitable for our aims. Therefore our goal is to find an implicit formula for the Schwarz operator \( T \) for our domain \( G(t) \) which will allow us to get the necessary properties of the solution.
Let us suppose that the operator $T$ is already constructed. Then the equation (11) can be rewritten in the following equivalent form

$$\frac{\partial f}{\partial t}(z, t) = z \frac{\partial f}{\partial z}(z, t) T \left( \left| \frac{\partial f}{\partial z} \right|^{-2} Q_j(t) \right), \quad z \in G(t), j = 1, 2. \quad (12)$$

For each fixed domain $G(t)$ we introduce the space $H(G(t)) \cap C^{1, \alpha}(G(t))$, as well as the spaces $H(G(t)) \cap C^\alpha(G(t)), H(G(t)) \cap \mathcal{C}(G(t))$. Then the following lemma is an analog of Lemma 1 [30, p. 104] which can be proved by the change of the unknown function in the same manner as in [30].

**Lemma 1.** Let $f(z, t) \in C^1 \left( [0, a_0], H(G(t)) \cap C^{1, \alpha}(G(t)) \right)$ for each $t \in [0, a_0]$ be a univalent function in $G(t)$ solving in $G(t) \times (0, a_0)$ the problem (12), (5) (or equivalently (4)-(6)).

Then the function $\phi(z, t) \equiv \left( \frac{\partial f}{\partial z}(z, t) \right)^{-1} \in C^1 \left( [0, a_0], H(G(t)) \cap C^\alpha(G(t)) \right)$ is a solution to the problem

$$\frac{\partial \phi}{\partial t}(z, t) = z T_{G(t)}(\phi)(z, t) \frac{\partial \phi}{\partial z}(z, t) - \phi(z, t) \frac{\partial}{\partial z} \left( z T_{G(t)}(\phi)(z, t) \right), \quad (z, t) \in G(t) \times (0, a_0), \quad (13)$$

$$\phi(z, 0) = \phi_0(z) \equiv \left( \frac{\partial f_0}{\partial z}(z) \right)^{-1}, \quad z \in G(t),$$

where $\phi(z, t) \neq 0$, and $T_{G(t)}(\phi)$ denotes the nonlinear operator

$$T_{G(t)}(\phi) \equiv T \left( |\phi|^2 Q(t) \right). \quad (15)$$

Conversely, if $\phi(z, t) \in C^1 \left( [0, a_0], H(G(t)) \cap C^\alpha(G(t)) \right)$ is a solution to (13)-(14) such that $\phi(z, t) \neq 0$ in $G(t) \times [0, a_0]$. Then $f(z, t) = \int_0^z \frac{dz}{\phi(t, z)}$ belongs to $C^1 \left( [0, a_0], H(G(t)) \cap C^{1, \alpha}(G(t)) \right)$ and represents a locally univalent solution to the problem (12), (5) (or equivalently (4)-(6)) in $G(t) \times (0, a_0)$.

By this lemma we reduce the starting problem (4)-(6) to the problem (13)-(14) which is a scale-type problem. Thus it remains to find a way how to interpret the problem (13)-(14) as a special case of (7). In Sec. 5 we will prove the existence of the solution $\phi(z, t)$ to the problem (13)-(14) which imply the following properties of the solution $f(z, t)$:

a) the family $f(z, t)$ belongs to the spaces $H(G(t')) \cap C^{1, \alpha}(G(t'))$ in the domains

$$G(t') \equiv \{ z \in \mathbb{C} : |z - a_j| > r_j(t'), j = 1, 2; |a_1 - a_2| > r_1(t') + r_2(t') \},$$

which are situated in a neighbourhood of $G(t)$;

b) $f(z, t)$ is univalent for small $t$ in the above said neighbourhood.
3. Scale of Banach spaces

Let us introduce analogously to [29] the following space of holomorphic functions (cf. [35]). In what follows we will assume for determines that \( Q_1(t), Q_2(t) \) are negative in a right-sided neighbourhood of \( t = 0 \):

\[
Q_1(t), Q_2(t) < 0, \quad t \in [0, T).
\]

(16)

Such assumption corresponds to the shrinkage of both holes in the Hele-Shaw cell. All other situation can be considered analogously.

Let us fix constants \( r_{i,j}, i, j = 1, 2; r_j(0) > r_{1,j} > r_{2,j} > 0 \), a positive number \( b > 0 \) and introduce parameter \( s \in (0, 1) \). Denote by \( \mathcal{H}(G(s)) \) the space of functions, holomorphic in the following circular doubly connected domain:

\[
G(s) \equiv \{ z \in \mathbb{C} : |z - a_j| > r_{1,j} - s(1 - r_{2,j}), j = 1, 2 \}.
\]

Then we define the space

\[
\mathcal{B} := \left\{ g = g(z, t) \in \bigcup_{0 < s < 1} \mathcal{C} \left( [0, b(1 - s)), \mathcal{H}(G(s)) \cap \mathcal{C}^{1, \alpha}(\partial G(s)) \right) : \right. 
\]

\[
\| g \|_{\mathcal{B}} = \max \left\{ \sup_{s \in (0, 1), h < b(1 - s)} \max_{t \in [0, h]} \| g(\cdot, t) \|_{\mathcal{C}^{\alpha}(\partial G(s))}, \right. 
\]

\[
\left. \sup_{s \in (0, 1), h < b(1 - s)} \left\| \frac{\partial g(\cdot, t)}{\partial z} \right\|_{\mathcal{C}^{\alpha}(\partial G(s))} \left( 1 - \frac{t}{b(1 - s)} \right)^{1/2} < \infty \right\}.
\]

The following lemmas are simply generalization of [29, Lemma 1, Lemma 2].

**Lemma 2.** The function space \( \mathcal{B} \) is a Banach space.

*Proof.* Let \( \{g_k\} \) be a Cauchy sequence from \( \mathcal{B} \). Then it is a Cauchy sequence in the spaces \( \mathcal{C} \left( [0, h], \mathcal{H}(G(s)) \cap \mathcal{C}^{1, \alpha}(\partial G(s)) \right) \) for each \( h < b(1 - s) \). It follows from the completeness of every of the latter spaces that there exists a uniquely determined limit element \( g = g(h, s) \in \mathcal{C} \left( [0, h], \mathcal{H}(G(s)) \cap \mathcal{C}^{1, \alpha}(\partial G(s)) \right) \). This element satisfies the estimate \( \max_{t \in [0, h]} \| g(h, s) \|_{\mathcal{C}^{\alpha}(\partial G(s))} \leq \| g_{k_0} \|_{\mathcal{B}} + \varepsilon \) for a suitable index \( k_0 \) uniformly for all \( h < b(1 - s) \) and \( s \in (0, 1) \). The limit element of our Cauchy sequence \( \{g_k\} \) is the function \( g = g(z, t) \) with the property that \( g \) coincides with \( g(h, s) \) for all \( h < b(1 - s), s \in (0, 1) \). Consequently

\[
\sup_{s \in (0, 1), h < b(1 - s)} \max_{t \in [0, h]} \| g \|_{\mathcal{C}^{\alpha}(\partial G(s))} < +\infty.
\]

If \( h < b(1 - s) \) for a certain \( s \in (0, 1) \), then there exists a larger number \( p \in (s, 1) \) such that \( h < b(1 - p) \), too. Using Cauchy’s integral formula for \( g \) in \( \partial G(p) \)

\[
g(z, t) = -\frac{1}{2\pi i} \int_{\Gamma_1(p)} \frac{g(\tau, t)}{\tau - z} d\tau - \frac{1}{2\pi i} \int_{\Gamma_2(p)} \frac{g(\tau, t)}{\tau - z} d\tau + C,
\]

where \( \Gamma_1(p), \Gamma_2(p) \) are boundary circles of the domain \( G(p) \), \( C \) is a constant, and estimating \( \frac{\partial g}{\partial z} \) in \( \partial G(s) \) gives

\[
g \in \mathcal{C} \left( [0, b(1 - s)), \mathcal{H}(G(s)) \cap \mathcal{C}^{1, \alpha}(\partial G(s)) \right).
\]
Finally,
\[
\left\| \frac{\partial g}{\partial z} \right\|_{C^\alpha(G(s))} \left( 1 - \frac{t}{b(1-s)} \right)^{\frac{1}{2}} \leq \left\| \frac{\partial g_k_0}{\partial z} \right\|_{C^\alpha(G(s))} \left( 1 - \frac{t}{b(1-s)} \right)^{\frac{1}{2}} + \varepsilon
\]
\[
\leq \|g_k_0\|_B + \varepsilon
\]
gives the property
\[
\sup_{s \in (0,1), t < b(1-s)} \left\| \frac{\partial g}{\partial z}(\cdot, t) \right\|_{C^\alpha(G(s))} \left( 1 - \frac{t}{b(1-s)} \right)^{\frac{1}{2}} < \infty,
\]
i.e. \( g \in \mathcal{B} \).

**Lemma 3.** The function space \( \mathcal{B} \) is an algebra with
\[
\|g \cdot h\|_B \leq 2\|g\|_B\|h\|_B \quad \text{for all } g, h \in \mathcal{B}.
\]

**Proof.** We know that \( g, h \in C([0, b(1-s)), \mathcal{H}(G(s)) \cap C^1,\alpha(G(s))) \).

The statement follows from inequalities
\[
\left\| \frac{\partial (g \cdot h)}{\partial z} \right\|_{C^\alpha(G(s))} \leq \left\| \frac{\partial g}{\partial z} \right\|_{C^\alpha(G(s))} \cdot \left\| h \right\|_{C^\alpha(G(s))},
\]
\[
\left\| \frac{\partial}{\partial z} (g \cdot h) \right\|_{C^\alpha(G(s))} \leq \left\| \frac{\partial g}{\partial z} \right\|_{C^\alpha(G(s))} \cdot \left\| h \right\|_{C^\alpha(G(s))} + \left\| g \right\|_{C^\alpha(G(s))} \left\| \frac{\partial}{\partial z} h \right\|_{C^\alpha(G(s))},
\]
and the definition of \( \|g\|_B \) and \( \|h\|_B \).

In the case of simply connected domains the pre-image of the domains occupied by one of the fluids in the Hele-Shaw can be fixed (e.g. the unit disc). It is not the case when the domain is doubly connected. For each fixed \( t \) one can choose two discs as pre-images of the “holes” in the Hele-Shaw cell. With time changing the ratio of the radii of these discs is changing too (one can suppose that one of these radius is constant but another one is changing; alternatively, both radii can be supposed changing).

Later in the paper we need also the following result:

**Lemma 4.** Let
\[
\mathcal{Z} \equiv \left\{ \zeta \in C^1(\mathbb{T}, \mathbb{C}) : \inf \left\{ \left| \frac{\zeta(s) - \zeta(t)}{s - t} \right| : s, t \in \mathbb{T}, s \neq t \right\} > 0 \right\}
\]
be the set of simple curves \( \zeta \) of class \( C^1(\mathbb{T}, \mathbb{C}) \).

Then the operator which assigns to each pair of simple smooth curves \( (\zeta^1, \zeta^2) \) from
\[
\mathcal{Z} \equiv \left\{ \zeta \equiv (\zeta^1, \zeta^2) \in \mathcal{Z}^2 : \zeta^1(\mathbb{T}) \subset \mathbb{E}[\zeta^2], \zeta^2(\mathbb{T}) \subset \mathbb{E}[\zeta^1] \right\}
\]
the conformal modulus of the corresponding doubly connected domain is real analytic locally near any fixed pair \((\zeta^1_0, \zeta^2_0) \in \mathcal{Z}\).

The proof in the case of annulus is given in [18, Theorem 6.1] on the base of the Implicit Function Theorem. The proof in the case of any other doubly connected domain follows from conformal equivalence under Möbius transformation.
4. Schwarz alternation method

In order to get the corresponding properties of the solution to the Schwarz problem we introduce the following notations and results following mainly to [19]. We formulate them in the case of an arbitrary circular \( n \)-connected domain.

Let \( \mathbb{D} \equiv \hat{\mathbb{C}} \setminus \left( \bigcup_{k=1}^{n} \text{cl} \mathbb{D}_k \right) \) be a circular plane multiply connected domain, \( \mathbb{D}_k \equiv \{ z \in \mathbb{C} : |z - a_k| < r_k \} \). Let the formula

\[
 z^{(k_m, k_{m-1}, \ldots, k_1)} = \left( z^{(k_{m-1}, \ldots, k_1)} \right)^*_{(k_m)} \tag{17} \]

defines successive symmetries with respect to the circles \( T_k = \partial \mathbb{D}_k \), \( k = 1, \ldots, m \). If in the sequence \( k_m, k_{m-1}, \ldots, k_1 \) no two neighbouring indexes are equal then the number \( m \) is called the level of the mapping \( z \mapsto z^{(k_m, k_{m-1}, \ldots, k_1)} \). When \( m \) is even, these mappings are Möbius transformations. When \( m \) is odd, these mappings are anti-Möbius transformations, i.e. Möbius with respect to \( z \). The above described successive symmetries forms for each fixed domain \( \mathbb{D} \) so called Schottky group \( K \) of symmetries (see [9]). In the following we denote by \( G \) the subgroup of all but not identity elements of \( K \) of even level, and by \( F \) the family of all elements of \( K \) of odd level.

The following theorems represent the solution \( \Psi(z) \equiv T(\mathbb{D}, q)(z) \) to the Schwarz problem

\[
 \begin{align*}
 \text{Re} \Psi(t) &= q(t), \ t \in \partial \mathbb{D}, \\
 \text{Im} \Psi(z_0) &= 0
\end{align*} \tag{18} \]

for a multiply connected circular domain \( \mathbb{D} \) in the above introduced notations, where \( q \equiv (q_1, \ldots, q_n) \), \( q_k : \mathbb{T}_k \to \mathbb{C} \) are given real-valued functions, \( z_0 \in \mathbb{D} \) is a given point inside \( \mathbb{D} \). We denote by \( T(\mathbb{D}, q)(\cdot) : q \mapsto \Psi \) the Schwarz operator of the domain \( \mathbb{D} \) with density \( q \).

**Theorem 2.** [19, Theorem 4.11] The Schwarz operator \( T(\mathbb{D}, q)(\cdot) \) of a multiply connected circular domain \( \mathbb{D} \equiv \hat{\mathbb{C}} \setminus \left( \bigcup_{k=1}^{n} \text{cl} \mathbb{D}_k \right) \) has the form

\[
 T(\mathbb{D}, q)(z) \equiv \Psi(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\mathbb{T}_k} q_k(\zeta) \left\{ \sum_{j} \left[ \frac{1}{\zeta - \gamma_j(w)} - \frac{1}{\zeta - \gamma_j(z)} \right] \right\} \\
 + \left( \frac{r_k}{\zeta - a_k} \right)^2 \sum_{j} \left[ \frac{1}{\zeta - \gamma_j(z)} - \frac{1}{\zeta - \gamma_j(w)} \right] - \frac{1}{\zeta - z} d\zeta \tag{19} \]

\[
 + \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\mathbb{T}_k} q_k(\zeta) \frac{\partial A}{\partial \nu}(\zeta) d\sigma + \sum_{m=1}^{n} A_m [\log (z - a_m) + \psi_m(z)] + i\varsigma,
\]

where

\[
 A_m := \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\mathbb{T}_k} q_k(\zeta) \frac{\partial \alpha_m}{\partial \nu}(\zeta) d\sigma, \ m = 1, 2, \ldots, n,
\]
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\[ A(\zeta) = \sum_{m=1}^{n-1} \text{Re} \left[ \phi_m \left( w_n^* \right)_m - \phi_m \left( w_m^* \right) \right] + \log |\zeta - w_n^*| - \alpha_n(\zeta) \log r_n - \sum_{m=1}^{n-1} \alpha_m(\zeta) \log |w_n^* - a_m|, \]  

(20)

the harmonic measure corresponding to \( m \)-th circle \( \alpha_m(\zeta) \) has the form

\[ \alpha_l(z) = \sum_{m=1, m \neq l}^n A_m [\text{Re} \psi_m(z) + \log |z - a_m|] + A, \]  

(21)

\( \psi_m(z) \) are derived in

\[ \psi_m(z) = \log \left[ \prod_{j=1, j \neq m}^{\infty} \psi_j^m(z) \right], \]  

(22)

where

\[ \psi_j^m(z) = \begin{cases} \frac{\gamma_j(z) - a_m}{\gamma_j(w) - a_m}, & \text{if } \gamma_j \in G, \\ \frac{\gamma_j(w) - a_m}{\gamma_j(z) - a_m}, & \text{if } \gamma_j \in F. \end{cases} \]

\( \zeta \) is an arbitrary real constant, \( \sum' \) contains \( \gamma_j \) of odd level (\( \gamma_j \in F \)), and \( \sum'' \) of even level (\( \gamma_j \in G \)). The series converges uniformly in each compact subset of \( \text{cl } \mathbb{D} \backslash \{\infty\} \).

The single-valued part of the Schwarz operator appears in the modified Dirichlet problem:

\[ \text{Re } \Psi (s) = q_k(s) + c_k, \quad t \in \mathbb{T}_k, \quad k = 1, 2, \ldots, n, \]  

(23)

where a given function \( q_k \in C(\partial \mathbb{D}_k), \) \( k = 1, \ldots, n \), \( c_k \) are undetermined real constants. If one of the constants \( c_k \) is fixed arbitrary, then the remaining ones are determined uniquely and \( \Psi(z) \) is determined up to an arbitrary additive purely imaginary constant. Thus we have

**Theorem 3.** [19, Theorem 4.12] The single-valued part of the Schwarz operator \( T_{single}(\mathbb{D}, q) \) of a multiply connected circular domain \( \mathbb{D} \equiv \mathbb{C} \backslash \left( \bigcup_{k=1}^n \text{cl } \mathbb{D}_k \right) \) corresponding to the modified Dirichlet problem (23) has the form

\[ T_{single}(\mathbb{D}, q)(z) \equiv \Psi(z) = \left\{ \begin{array}{ll}
\frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\mathbb{T}_k} (q_k(\zeta) + c_k) \left\{ \sum_{j}'' \left[ \frac{1}{\zeta - \gamma_j(w)} - \frac{1}{\zeta - \gamma_j(z)} \right] + \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\mathbb{T}_k} q_k(\zeta) \frac{\partial A}{\partial \nu}(\zeta) d\sigma + i\varsigma. \end{array} \right. \]  

(24)
One of the real constants $c_k$ can be fixed arbitrarily; the remaining ones are determined uniquely from the linear algebraic system

$$
\sum_{k=1}^{n} \int_{\Gamma_k} (q_k(\zeta) + c_k) \frac{\partial \alpha_m(\zeta)}{\partial \nu} d\sigma = 0, \quad m = 1, 2, \ldots, n - 1. \tag{25}
$$

By applying the properties of the Cauchy type operators (see e.g. [10]) and the successive symmetries as above introduced (see e.g. [19]) one can establish the following result

**Theorem 4.** The Schwarz operator $T(\mathbb{D}, q)(\cdot)$ (as well as the single-valued Schwarz operator $T_{\text{single}}(\mathbb{D}, q)(\cdot)$) of a multiply connected circular domain $\mathbb{D} \equiv \hat{\mathbb{C}} \setminus \left( \bigcup_{k=1}^{n} \text{cl} \, D_k \right)$ is a bounded operator from $\prod_{k=1}^{n} \mathcal{C}^{m,\alpha}(\mathbb{T}_k, \mathbb{R})$ to $\prod_{k=1}^{n} \mathcal{H}(\text{ext} \, \mathbb{T}_k, \mathbb{C}) \cap \mathcal{C}^{m,\alpha}(\mathbb{T}_k, \mathbb{C})$.

**Proof.** From the induction argument follows that it suffices to prove the statement in the case of a doubly connected domain (i.e. for $n = 2$) and for $m = 1$. The above Theorems 2, 3 give then the existence of the solution to corresponding problem. The form of the Schwarz operators (see (19), (24)) does not allow to obtain boundedness of these operators as well as their regularity. Thus we use an another approach.

Let us represent the solutions to the problem

$$
\text{Re } \Psi(s) = q_k(s), \quad t \in \mathbb{T}_k, \quad k = 1, 2, \tag{26}
$$

in the following form

$$
\Psi(z) = \Psi_1(z) + \Psi_2(z), \quad z \in \text{cl} \, \mathbb{D}, \tag{27}
$$

where the functions $\Psi_k, \quad k = 1, 2$, are holomorphic in the domains $\text{ext} \, \mathbb{T}_k \equiv \{ z \in \mathbb{C} : |z - a_k| > r_k \}$, continuous in $\text{cl} \, \text{ext} \, \mathbb{T}_k \equiv \{ z \in \mathbb{C} : |z - a_k| \geq r_k \}$. Then the first equation of (26) can be rewritten as

$$
\text{Re } \Psi_1(s) + \text{Re } \Psi_2(s^*_{(1)}) = q_1(s), \quad |s - a_1| = r_1, \tag{28}
$$

where $s^*_{(1)}$ is the point symmetric to $s$ with respect to the first circle $\mathbb{T}_1$ (thus $s^*_{(1)} = s$).

By applying to the boundary condition (28) the Schwarz operator of the exterior of the first circle, which is determined uniquely up to the real constant $\zeta_1$ by the following formula

$$
T_1 q_1(z) \equiv T(\text{ext} \, \mathbb{T}_1, q_1)(z) \equiv -\frac{1}{2\pi i} \int_{|\tau - a_1| = r_1} \frac{\tau + z - 2a_1 q_1(\tau) d\tau}{\tau - z} \tau - a_1 + i\zeta_1, \tag{29}
$$

we arrive at the functional equation in the domain $\text{ext} \, \mathbb{T}_1$

$$
\Psi_1(z) = -\Psi_2(z^*_{(1)}) + T_1 q_1(z), \quad |z - a_1| \geq r_1. \tag{30}
$$

Analogously we obtain the functional equation in the domain $\text{ext} \, \mathbb{T}_2$

$$
\Psi_2(z) = -\Psi_1(z^*_{(2)}) + T_2 q_2(z), \quad |z - a_2| \geq r_2, \tag{31}
$$

where operator $T_2$ is defined by the formula

$$
T_2 q_2(z) \equiv T(\text{ext} \, \mathbb{T}_2, q_2)(z) \equiv -\frac{1}{2\pi i} \int_{|\tau - a_2| = r_2} \frac{\tau + z - 2a_2 q_2(\tau) d\tau}{\tau - z} \tau - a_2 + i\zeta_2, \tag{32}
$$
The relations (30), (31) constitute a system of functional equations in a class of holomorphic functions. The system (30), (31) can be solved by successive approximations. Let us consider now the method of successive approximations for (30), (31) from another point of view. We take zero-order approximation for (30), (31) in the form
\[ \Psi^0(z) \equiv T_1 q_1(z), \quad |z - a_1| \geq r_1. \]

The function \( \Psi^0(z) \) satisfies the first boundary condition of (26), but possibly does not satisfy the second one. This function is defined and holomorphic in the exterior of the first circle. Thus we can take it as a zero-order approximation to the solution in the domain \( D \equiv \{ z \in \mathbb{C} : |z - a_k| > r_k, \ k = 1, 2 \} \), namely
\[ \Psi^0(z) \equiv \Psi^0(z)_{|D}, \quad z \in D. \]

Substituting \( \Psi^0(z) \) into the functional equation (31) we obtain the first order
\[ \Psi^1(z) \equiv -\Psi^0(z^*_2) + T_2 q_2(z), \quad |z - a_2| \geq r_2, \]

and respectively
\[ \Psi^1(z) \equiv \Psi^0(z) + \Psi^2(z)_{|D}, \quad z \in D. \]

The last function does satisfy the second boundary condition of (26), but possibly does not the first one. Further
\[ \Psi^1(z) \equiv -\Psi^1(z^*_1), \quad |z - a_1| \geq r_1, \]

and consequently
\[ \Psi^2(z) \equiv \Psi^0(z) + \Psi^1(z) + \Psi^2(z)_{|D}, \quad z \in D. \]

Finally, an \( N \)-th approximation of the solution can be given by the following formula:
\[ \Psi^N(z) \equiv \Psi^0(z) + \Psi^1(z) + \ldots + \Psi^{N-1}(z) + \Psi_N(z)_{|D}, \quad z \in D, \ N \text{ is even number}, \]
\[ \Psi^N(z) \equiv \Psi^0(z) + \Psi^1(z) + \ldots + \Psi^{N-1}(z) + \Psi^N(z)_{|D}, \quad z \in D, \ N \text{ is odd number}. \]

The proof of the convergence of the sequence \( \Psi^N(z) \) in the space \( \mathcal{H}(D) \cap C^{1,\alpha}(\text{cl}D) \) for each fixed pair of functions \( q \equiv (q_1, q_2) \in C^{0,\alpha}(T_1) \times C^{0,\alpha}(T_1) \) is presented in [19, Sec. 4.9]. The similar proof can be given in the case of the space \( \mathcal{H}(D) \cap C^{1,\alpha}(\text{cl}D) \) for a fixed pair of data \( q \equiv (q_1, q_2) \in C^{1,\alpha}(T_1) \times C^{1,\alpha}(T_1) \).

In order to show it we have to mention that the solution of the Schwarz problem for a doubly connected circular domain \( D \) can be rewritten in the form of sum of two series
\[ \Psi(z) \equiv \Psi^1(z) + \Psi^2(z), \]

where
\[ \Psi^1(z) \equiv T_1 q_1(z) + T_1 q_1(\alpha(z)) + T_1 q_1(\alpha^2(z)) + \ldots + T_1 q_1(\alpha^n(z)) + \ldots, \]
\[ \Psi^2(z) \equiv T_2 q_2(z^*_1) + T_2 q_2(\beta(z)) + T_2 q_2(\beta^2(z)) + \ldots + T_2 q_2(\beta^n(z)) + \ldots, \]
\[ \alpha(z) \equiv z^*_{(1)}, \quad \beta(z) \equiv z^*_{(2)}, \quad \alpha^n(z) \equiv \alpha(\alpha^{n-1}(z)), \quad \beta^n(z) \equiv \beta(\beta^{n-1}(z)). \]

Both operators \( \alpha : z \mapsto \alpha(z), \beta : z \mapsto \beta(z) \) are compact in the space \( \mathcal{H}(D) \cap C^{1,\alpha}(\text{cl}D) \) (see [19, Subsec. 4.9.3]). Hence it remains to prove that operators \( T_1, T_2 \) are bounded as operators acting from \( C^{1,\alpha}(T_1), C^{1,\alpha}(T_2) \) to \( \mathcal{H}(D) \cap C^{1,\alpha}(\text{cl}D) \) respectively. This
result is basically a restatement of two technical lemmas from [18, Lemmas 5.1, 5.2]. It completes the proof. □

5. APPLICATION OF THE ABSTRACT CAUCHY-KOVALEVSKY PROBLEM

In this Section we apply Nirenberg-Nishida theorem (Theorem 1) to the abstract Cauchy-Kovalevsky problem (13)-(14), where the operator $T_{G(t)}$ is defined by the formula (15). We follow here the ideas and machinery of the papers [30], [29]. We can formulate the main result of our paper in the following manner.

**Theorem 5.** [Main Theorem] There exists an in general small interval of time $[0, b)$ such that the problem (13)-(14) has a uniquely determined solution $\phi = \phi(z, t)$ which has no zeros on $G(p) \times [0, b)$ and belongs to the space $C^1 \left([0, b), \mathcal{H}(G(p)) \cap C^{1,\alpha}(G(p))\right)$. The constant $p$ determined the doubly connected domain $G(p)$ is taken as in the definition of $B$.

**Corollary 1.** There exists an in general small interval of time $[0, b)$ such that the problem (4)-(6) has a uniquely determined solution $f$. The function $f = f(z, t)$ is univalent with respect to $z$ on $G(p)$ for each $t \in [0, b)$ and belongs to the space $C^1 \left([0, b), \mathcal{H}(G(p)) \cap C^{2,\alpha}(G(p))\right)$. The constant $p$ determined the doubly connected domain $G(p)$ is taken as in the definition of $B$.

This corollary is a direct consequence of the theorem. Univalence of the function $f$ follows from the properties of initial function $f_0$, the choice of $r_{i,j} i, j = 1, 2; r_j(0) > r_{1,j} > r_{2,j} > 0$ (see definition of $B$) and the chosen norm of $B$ which is stronger than the sup-norm.

To prove the main theorem we need a number of auxiliary results which we present below. We apply Theorem 1 to the problem (13)-(14). Hence the operator $L(t, \phi)$ coincides with the operator in the right-hand side of (13). Thus the operator $L(t, \phi)$ contains mainly the combination of two operators: differentiation operator $\partial z$ and the Schwarz operator $T$ for a doubly connected circular domain applied to $|\phi|^2$.

To study the unbounded operator $\partial z$ in the scale $B$ we use the following result which is based on the idea formulated in [37]:

**Lemma 5.** Let $\phi \in B$. Then the operator

$$A : \phi \mapsto \int_0^t \partial z \phi(\cdot, \tau) d\tau, \quad t \in [0, b(1 - s)), \quad s \in (0, 1),$$

is a continuous operator mapping $B$ into itself, and satisfying the estimate

$$\|A\phi\|_B \leq Cb\|\phi\|_B, \quad \text{(36)}$$

where $C = C(r_{i,j}), i, j = 1, 2$.

This lemma is a restatement of [29, Lemma 3].

Let us now begin to establish the necessary properties of the operator $T_{G(t)}(\phi) \equiv T \left(|\phi|^2 Q_j(t)\right)$. In the previous section we have shown the boundedness of this operator in the corresponding spaces for each fixed domain $G(t)$. But we need in Theorem 1 the fulfillment of stronger conditions.
First of all, the function \( T \left( |\phi|^2 Q_j(t) \right)(z) \) have to be continued analytically in a neighbourhood of the initial doubly connected domain \( G(0) \) (thus to preserve the analogous property of the “initial” function \( \phi_0 \)). To see it we mention first that the Schwarz operator can be rewritten in the form
\[
T \left( |\phi|^2 Q_j(t) \right)(z) = T \left( \phi(\tau, t) \phi \left( \frac{1}{r}, t \right) Q_j(t) \right)(z).
\]
(37)

If the function \( \phi(z, t) \) can be continued in the domain bigger than \( G(t) \) then the conjugate function \( \tilde{\phi}(z, t) \equiv \overline{\phi \left( \frac{1}{\bar{r}}, t \right)} \) possesses the “continuation” in the smaller domain. Therefore we introduce another function space different from \( B \).

We fix as before the constants \( r_{i,j}, i, j = 1, 2; r_j(0) > r_{1,j} > r_{2,j} > 0 \), a positive number \( b > 0 \) and introduce parameter \( s \in (0, 1) \). Denote by \( \mathcal{H}(A(s)) \) the space of functions, holomorphic in the union of two annulus:
\[
A(s) \equiv A^1(s) \bigcup A^2(s),
\]
\[
A^j(s) \equiv \left\{ z \in \mathbb{C} : r_{1,j} - s(r_{1,j} - r_{2,j}) < |z - a_j| < \frac{1}{r_{1,j} - s(r_{1,j} - r_{2,j})} \right\}, j = 1, 2.
\]
Then we define the space
\[
\mathcal{B}_a := \left\{ g = g(z, t) \in \bigcup_{0<s<1} C \left( [0, b(1-s)), \mathcal{H}(A(s)) \cap C^{1,\alpha}(\overline{A(s)}) \right) : \right. \\
\left. \|g\|_\mathcal{B} = \max \left\{ \sup_{s \in (0,1), h < b(1-s)} \max_{t \in [h,b]} \|g(\cdot, t)\|_{C^{0}(\overline{A(s)})} ; \\
\sup_{s \in (0,1), t < b(1-s)} \left\| \frac{\partial g(\cdot, t)}{\partial z} \right\|_{C^{0}(\overline{A(s)})} \left( 1 - \frac{t}{b(1-s)} \right)^{\frac{j}{2}} \right\} < \infty \right\}.
\]

It is not hard to see that the space \( \mathcal{B}_a \) is a Banach space and Banach algebra with respect to usual multiplication of complex-valued functions. The following lemma is of the straightforward verification.

**Lemma 6.** If \( \phi \) belongs to \( \mathcal{B} \), then the function \( \tilde{\phi}(z, t) \equiv \overline{\phi \left( \frac{1}{\bar{r}}, t \right)} \) as well as the product \( \tilde{\phi} \phi \) belongs to \( \mathcal{B}_a \). Besides,
\[
\|\tilde{\phi}\|_{\mathcal{B}_a} \leq C\|\phi\|_{\mathcal{B}}.
\]

Now all is prepared for the study of \( T_{G(t)}(\phi) \) on \( B \). Let
\[
G_{\eta_1, \eta_2} \equiv \{ z \in \mathbb{C} : |z - a_1| > \eta_1 r_1, \ |z - a_2| > \eta_2 r_2 \}
\]
be a neighbourhood of a doubly connected domain \( G(0) \). Let the function \( v \) be in \( C \left( [0, T), \mathcal{H}(G_{\eta_1, \eta_2}) \cap C(\overline{G_{\eta_1, \eta_2}}) \right) \) for certain \( \eta_1, \eta_2 < 1 \). Then the function \( T_{G(t)}(v) \equiv T(G(t), q|v|^2) \) as defined in (15) belongs to \( \mathcal{H}(G(t)) \) for each \( t \in [0, T) \). Moreover, the following lemma is valid.

**Lemma 7.** For any \( v \in C \left( [0, T), \mathcal{H}(G_{\eta_1, \eta_2}) \cap C(\overline{G_{\eta_1, \eta_2}}) \right) \) the image \( T_{G(t)}(v) \) of the nonlinear operator \( T_{G(t)} \) possesses an analytic continuation into the bigger domain \( G_{\eta_1, \eta_2} \) for certain \( \eta_1^0 \leq \eta_1 < 1, \ \eta_2^0 \leq \eta_2 < 1 \).
Lemma 8. The operator

\[ T_{G(t)}(\phi) \equiv T \left( |\phi|^2 Q(t) \right) \]

satisfies in \( B \) the following inequalities

\[ \left\| T_{G_{q_1} q_2}(\phi) \right\|_B \leq C_1 \|\phi\|^2_B, \]

\[ \left\| T_{G_{q_1} q_2}(\phi_1) - T_{G_{q_1} q_2}(\phi_2) \right\|_B \leq C_1 \|\phi_1 - \phi_2\|^2_B. \]

Moreover the family of operators \( \{T_{G(t)}(\phi)\}_{t \in [0,T]} \) depends continuously on \( t \in [0,T] \).

Proof. We have to start again with the operator \( T_1 \). By rewriting it in the form (38) we can use then [29, Lemma 5]. Final result follows then from the properties of series \( \tilde{\Psi}_1(z), \tilde{\Psi}_2(z) \).

At last we need to show that the right-hand side of (13) does satisfy the conditions of Theorem 1. To do it we rewrite equations (13) in suitable form

\[ \frac{\partial \omega}{\partial t}(z, t) = L_0(t, \omega) = z T_{G_{q_1} q_2}(\omega + \phi_0)(z, t) \frac{\partial}{\partial z}(\omega(z, t) + \phi_0(z)) \]

\[ - (\omega(z, t) + \phi_0(z)) \frac{\partial}{\partial z} \left( z T_{G_{q_1} q_2}(\omega + \phi_0)(z, t) \right), \]

\[ \omega(z, 0) = \phi(z, 0) - \phi_0(z) = 0. \]
Lemma 9. The operator $L_0$ satisfies in the scale $\{B_s, \| \cdot \|_s \}_{0<s \leq 1}$ of Banach spaces,

$$\{B_s\}_{0<s \leq 1} := \left\{ g = g(z, t) \in \mathcal{C} \left( [0, b(1-s)], \mathcal{H}(G(s)) \cap \mathcal{C}^{1,\alpha}(\overline{G(s)}) \right) : \right.$$  

$$\left\| g \right\|_s = \max \left\{ \sup_{s \in (0,1), h < b(1-s)} \max_{t \in [0,h]} \| g(\cdot, t) \|_{\mathcal{C}^{\alpha}(\overline{G(s)})} ; \right.$$  

$$\sup_{s \in (0,1), t < b(1-s)} \left\| \frac{\partial g(\cdot, t)}{\partial z} \right\|_{\mathcal{C}^{\alpha}(\overline{G(s)})} \left( 1 - \frac{t}{b(1-s)} \right)^{\frac{1}{2}} < \infty, \right\}.$$  

the conditions (8)-(10) of Theorem 1.

Proof. The property (8) follows from the second statement of Lemma 8. Further, Lemma 5 gives that the operator $\frac{\partial}{\partial z}$ is bounded operator from $B_s$ to $B_{s'}$ and

$$\left\| \frac{\partial}{\partial z} \right\|_{s \rightarrow s'} \leq \frac{1}{\min\{r_{1,1} - r_{2,1}; r_{1,2} - r_{2,2}\}(s - s')}.$$  

(41)  

In order to get the properties (9)-(10) we rewrite the difference $L_0(t, \omega_1) - L_0(t, \omega_2)$ in the following form:

$$L_0(t, \omega_1) - L_0(t, \omega_2) \equiv - (\omega_1 - \omega_2) \frac{\partial}{\partial z} \left( z T_{G_{\eta_1, \eta_2}}(\omega_1 + \phi_0)(z, t) \right) -$$  

$$- (\omega_2 + \phi_0) \frac{\partial}{\partial z} \left( z T_{G_{\eta_1, \eta_2}}(\omega_1 + \phi_0)(z, t) - T_{G_{\eta_1, \eta_2}}(\omega_2 + \phi_0)(z, t) \right) +$$  

$$+ z (T_{G_{\eta_1, \eta_2}}(\omega_1 + \phi_0)(z, t) - T_{G_{\eta_1, \eta_2}}(\omega_2 + \phi_0)(z, t)) \frac{\partial}{\partial z}(\omega_1 + \phi_0) +$$  

$$+ z T_{G_{\eta_1, \eta_2}}(\omega_2 + \phi_0)(z, t) \frac{\partial}{\partial z}(\omega_1 - \omega_2).$$

Then inequality (9) follows from Lemma 8 and inequality (41).

In the same manner we can prove that

$$\left\| L_0(t, 0) \right\|_s = \left\| \phi_0 \frac{\partial}{\partial z} \left( z T_{G_{\eta_1, \eta_2}}(\phi_0)(z, t) \right) - z T_{G_{\eta_1, \eta_2}}(\phi_0)(z, t) \frac{\partial}{\partial z} \phi_0(z) \right\|_s \leq \frac{K}{1-s}.$$  

□

By applying Lemma 9 we obtain the existence and uniqueness of the local (in time) analytic solution to the Hele-Shaw for a doubly connected domain.

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References


