On Line Graphs of Linear 3-Uniform Hypergraphs

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Abstract: It is known that the class of line graphs of linear 3-uniform hypergraphs cannot be characterized by a finite list of forbidden induced subgraphs (R. N. Naik, S. B. Rao, S. S. Shrikhande, and N. M. Singhi, Intersection Graphs of k-uniform linear hypergraphs, *Eur. J. Combin.* 3 (1982), 159–172). In this paper such a characterization is given provided that no vertex has degree less than 19. An analogous characterization for graphs whose vertex degrees are not less than 69 was obtained in Naik et al. © 1997 John Wiley & Sons, Inc. J Graph Theory 25: 243–251, 1997

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1. INTRODUCTION

The following two characterizations of line graphs of simple graphs are well known.

**Krausz Theorem** [4]. A graph $G$ is the line graph $L(H)$ of some simple graph $H$ if and only if there exists a clique covering of $G$ satisfying the following conditions:

(i) each vertex of $G$ belongs to at most two cliques of the covering,

(ii) any two cliques have at most one common vertex.

**Beineke Theorem** [1]. A graph $G$ is the line graph of some simple graph if and only if none of the nine graphs shown in Figure 1 is an induced subgraph of $G$.

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The line graph $L(H)$ of a hypergraph $H$ is the intersection graph of the edges of $H$. Thus, the vertices of $L(H)$ are in one-to-one correspondence with the edges of $H$, and two vertices are adjacent in $L(H)$ if and only if the corresponding edges are adjacent in $H$.

A hypergraph is called linear if any two edges have at most one common vertex. A hypergraph whose degree is all equal to $r$ is called $r$-uniform.

A finite family $C = \{C_1, C_2, \ldots, C_k\}$ of cliques of a graph $G$ is called a covering if $G$ is the union of these cliques; each clique $C_i$ is a component of the covering. A covering is called linear if any two of its components have at most one common vertex. A covering is called an $r$-covering if each vertex of a graph belongs to at most $r$ components.

**Theorem A** [5]. Any maximal $r$-large clique of a graph is a constituent of each linear $r$-covering.

**Theorem B** [7]. A graph belongs to the class $\mathcal{L}_r$ if and only if it has a linear $r$-covering.

**2. CHARACTERIZATION OF GRAPHS IN $\mathcal{L}_r$ WITH $\omega(G) \geq 19$**

Throughout the paper $V(G)$ and $E(G)$ are the vertex and edge sets of the graph $G$, respectively.

Define the sets $A_r = \{1, 2, \ldots, r\}$. $A_r$ is the set of graphs of order $2r$ having a dominating vertex (i.e., a vertex adjacent to each other one) and not containing $K_r$.

For an arbitrary graph belonging to $\mathcal{L}_r$, the list of maximal cliques can be constructed in polynomial time in the vertex number, the power of the polynomial increases together with $r$ [15]. Therefore, in some cases the minimum value $r$ such that $G \in \mathcal{L}_r$ would be accepted as a measure of complexity of a graph $G$. The recognition problem $\"G \in \mathcal{L}_r\"$ for fixed $r \geq 3$ is NP-complete [15]. Unfortunately, we do not know the complexity of the analogous problem for the class $\mathcal{L}_r$.

This paper is inspired by the paper [7] given a finite induced subgraph characterization of graphs in $\mathcal{L}_r$ whose vertex degrees are at least 69. In Section 2 of our paper, this degree bound is reduced to 19, and the list of subgraphs differs from the list in [7]. The proof is based on the theorems of Kruskal [14] and Brouwer [1], the Kruskal characterization of the class $\mathcal{L}_2$ [1] and properties of graph cliques from this class obtained in [5]. Using the same method and a simpler technique, we get an analogous characterization of graphs in $\mathcal{L}_r$ with vertex degrees at least $c r^2 + 3$ and $c$ is an arbitrary constant (Theorem 3.3). In fact, this was conjectured in [7].

Nevertheless, such a finite characterization was obtained in [7] with a restriction on edge degrees.

In Section 3 we describe a procedure for recognizing $\"G \in \mathcal{L}_r\"$ in the class of graphs with $\omega(G) \geq 19$ with polynomial complexity.

An arbitrary complete subgraph of a graph is called a clique. The set of vertices of a clique may be called a clique as well, without confusion. A clique $C$ is called $r$-large if $|C| \geq r^2 - r + 2$.

A finite family $C = \{C_1, C_2, \ldots, C_k\}$ of cliques of a graph $G$ is called a covering if $G$ is the union of these cliques; each clique $C_i$ is a component of the covering. A covering is called linear if any two of its components have at most one common vertex. A covering is called an $r$-covering if each vertex of a graph belongs to at most $r$ components.

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**Theorem A** [5]. Any maximal $r$-large clique of a graph $G$ is a constituent of each linear $r$-covering.

**Theorem B** [7]. A graph belongs to the class $\mathcal{L}_r$ if and only if it has a linear $r$-covering.
(f) \(8 \leq |G_i| \leq 9, \ i = 1, \ldots, p;\)
(g) each vertex of \(B\) belongs to one or two cliques \(G_i\), and each vertex of degree 2 belongs to two.

For each \(i, 1 \leq i \leq 8\), the orders of graphs in \(A_i\) are restricted. Hence \(A_i\) is finite. For example, \(|G| = 14\) for \(G \in A_4\). For \(G \in A_6\), conditions (d), (e) and (g) imply \(p \leq 12\).

Taking the union of \(A_i, 1 \leq i \leq 8\), we get a finite list \(A\).

**Theorem 3.1.** For a graph \(G\) with \(\delta(G) \geq 19\), the following two statements are equivalent:

(i) \(G \in L_3^1\);

(ii) none of the graphs from the list \(A\) is an induced subgraph of \(G\).

**Proof.** (i) implies (ii). Obviously, \(L_3^1\) is a hereditary class of graphs. Hence it is sufficient to prove that

\[ A \cap L_3^1 = \emptyset. \quad (1) \]

By Theorem A, the neighborhood of every vertex in a graph in \(L_3^1\) has to be divided into at most three cliques. Therefore \((A_1 \cup A_5) \cap L_3^1 = \emptyset\).

Suppose that

\[ G \in A_2 \cap L_3^1. \quad (2) \]

By Theorem B, there exists a linear 3-covering \(C\) of the graph \(G\). Further, \(G\) contains a maximal clique \(K_8\) which must be a constituent of \(C\) by Theorem A. It is obvious that such a covering \(C\) cannot exist, and consequently (2) is impossible.

In the same way, the cliques \(G_1\) and \(G_2\) must be constituents of every linear 3-covering for \(G \in A_3 \cup A_4\). This is impossible, since \(|V G_1 \cap V G_2| > 1\). Therefore

\[ (A_3 \cup A_4) \cap L_3^1 = \emptyset. \]

Suppose that

\[ G = B \cup G_1 \cup G_2 \cup \cdots \cup G_p \cup F \in A_6 \cap L_3^1. \quad (3) \]

Fix a linear 3-covering \(C\) for \(G\). By Theorem A, every clique \(G_i\) is a constituent of this covering and every vertex of \(G\) belongs to some constituent \(G_i\). After deleting the edges of each \(G_i\) and any resulting isolated vertices from \(G\), the graph \(G\) becomes a graph \(H\) and the covering \(C\) becomes a linear 2-covering for \(H\). By the Krausz theorem, \(H \in L_2^1\), contrary to the Beineke theorem, since \(B\) is an induced subgraph in \(H\). Thus (3) is impossible.

Finally, for \(G \in A_7 \cup A_8\), every clique \(G_i\) must belong to each linear 3-covering. Obviously, such a covering cannot exist, and hence \((A_7 \cup A_8) \cap L_3^1 = \emptyset\). Equality (1) is therefore proved.

(ii) implies (i). Suppose that \(C = \{C_i; i = 1, \ldots, k\}\) is the set of maximal 3-large cliques of \(G, V_i = VC_i\).
Proposition 1. Every vertex of $G$ belongs to some clique $C_i$.

Indeed, let

$$v \in VG, U \subseteq N(v), |U| = 19, H = G(U \cup \{v\}).$$

Then $|H| = 20$ and $v$ is a dominating vertex in $H$. Further, $H$ contains the 3-large clique $K_8$, since $H \not\subseteq A_1$.

Proposition 2. If $v \in VG \setminus V_i$, then $v$ has at most three neighbors in $V_i$.

This is obvious, since otherwise $G$ would have an induced subgraph in $A_2$.

In particular, Proposition 2 implies $|V_i \cap V_j| \leq 3$ for $i \neq j$.

Proposition 3. $|V_i \cap V_j| \leq 1, i \neq j$.

This follows immediately from Proposition 2, since if $2 \leq |V_i \cap V_j| \leq 3$, then $G(V_i \cup V_j)$ contains an induced subgraph in $A_3$ or $A_4$.

Proposition 4. If $v \in V_i \cap V_j$ for $i \neq j$, then $N_G[v] = V_i \cap V_j$ or $N_G[v] = V_i \cup V_j \cup F$, where $F$ is a clique such that $F \cap (V_i \cup V_j) = \{v\}$.

Suppose to the contrary that $av$ and $bv$ are edges of $G$ that do not belong to $C_i$ and $C_j$, and $a$ and $b$ are not adjacent. By Proposition 2, there are vertices $c \in V_i$ and $d \in V_j$ which are adjacent to neither $a$ nor $b$ nor each other. We have

$$G(\{v, a, b, c, d\}) = K_{1,4} \in A_5.$$ 

Proposition 4 implies, in particular, that each vertex of the graph $G$ belongs to at most three cliques $V_i$. If there are exactly three such cliques, then the union of these cliques coincides with $N_G[v]$.

Now delete from $G$ the edges of all cliques $C_i$ belonging to the set $C$. Delete also all isolated vertices if any appear. Denote the resulting graph by $H$.

Proposition 5. Each vertex of the graph $H$ belongs to one or two cliques $V_i$.

This follows from Propositions 1 and 4.

Proposition 6. $H \in L_2$.

It is sufficient to prove that $H$ does not contain an induced subgraph forbidden by the Beineke theorem (Fig. 1). Suppose to the contrary that $B$ is one of the Beineke graphs and $C_B = \{C_i; i = 1, \ldots, q\}$ is the set of cliques from the list $C$ such that $C_i$ contains vertices from $B$. Consider the set $U_i$ of vertices $u \in V_i$ satisfying one of the following conditions: $u \in VB; u$ belongs to two cliques from $C_B$. Propositions 3 and 5 imply that $q \leq 12$, since $|B| \leq 6$, hence $|U_i| \leq 11$; $i = 1, \ldots, q$. Consequently, for every $C_i$ from $C_B$ there is a 3-large clique $G'_i$ with $|G'_i| \leq 11$ containing the set $U_i$. Consider the induced subgraph

$$G' = G(VG'_1 \cup VG'_2 \cup \cdots \cup VG'_q).$$

Obviously, it can be represented in the form

$$G' = B \cup G'_1 \cup G'_2 \cup \cdots \cup G'_q \cup F,$$

where

$$VF \subseteq V(G'_1 \cup G'_2 \cup \cdots \cup G'_q), \quad EF \cap E(B \cup G'_1 \cup G'_2 \cup \cdots \cup G'_q) = \emptyset.$$ 

By Proposition 3, every clique $G'_i$ is maximal in $G'$, and $G' \in A_6$. 
Proposition 7. There exists a linear 2-covering $D$ of the graph $H$ such that $C \cup D$ is a linear 3-covering of $G$.

Indeed, by the Krausz theorem and Proposition 6, there exists a linear 2-covering $D = (D_i : i = 1, 2, \ldots, s)$ of the graph $H$. Without loss of generality, suppose that $D$ contains no one-vertex constituents. Taking Proposition 5 into account, we get that $E = C \cup D$ is a linear 4-covering. Moreover, $E$ is a linear 3-covering if we choose $D$ in a certain way. Indeed, suppose to the contrary that $E$ is not a linear 3-covering. This means that there is a vertex $a$ belonging to three cliques, two from $C$, say $C_1$ and $C_2$, and two from $D$, say $D_1$ and $D_2$. Further, let $ab_1 \in ED_1$ and $ab_2 \in ED_2$. By Proposition 4, the vertices $b_1$ and $b_2$ are adjacent in $G$.

If $b_1b_2 \not\in EH$, then this edge belongs to some third clique $C_3 \subseteq C$. Any of the three cliques $C_i, i = 1, 2, 3$, has at most one vertex in common with each of the remaining two cliques and contains at most two vertices from the set $\{a, b_1, b_2\}$. Therefore, each of the cliques $C_i$ contains an 8-vertex clique $G_i'$ such that

$$G(V G_1' \cup V G_2' \cup V G_3' \subseteq A_7),$$

a contradiction.

Thus $b_1b_2 \in EH$. So, the cliques $D_1$ and $D_2$ together belong to some maximal clique $K$ of $H$. If $|K| \geq 4$, then $K$ is a constituent of the covering $D$ by Theorem A. Therefore $|K| = 3$, $VD_i = \{a, b_i\}, i = 1, 2$ and the edge $b_1b_2$ belongs to some third constituent $D_3$ of $D$.

Note that $|D_3| \leq 3$. If not, suppose that $b_3$ and $b_4$ are new vertices from $D_3$, $B = H(\{a, b_1, b_2, b_3, b_4\})$, $C_B = \{C_1, C_2, \ldots, C_5\}$ is the set of cliques from the list $C$ such that $C_i$ contains vertices of the graph $B$. Using the same arguments as in the proof of Proposition 6, we get that each clique $C_i$ from $C_B$ contains a 3-large clique $G_i'$ with $|G_i'| \leq 9$ such that the induced subgraph $G(V G_1' \cup V G_2' \cup \cdots \cup V G_5')$ is a graph of type 1 from the list $A_8$.

Now let $VD_3 = \{b_1, b_2, b_3\}$. Then the vertex $b_3$ belongs to only one clique from the list $C$, for otherwise we can find in $G$ an induced subgraph of type 2 from $A_8$ as above. In the same way, if $H$ contains a new vertex $b_4$ adjacent to $b_3$, then, by definition of the covering $D$, this vertex is adjacent to neither $a$ nor $b_i, i = 1, 2$. So, an induced subgraph of type 3 from $A_8$ can be found.

Thus, $b_3$ belongs to exactly two constituents of the covering $E$, namely, to $D_3$ and one constituent from $C$. Remove from $D$ the constituents $D_i, i = 1, 2, 3$, and add the new ones $K_1, K_2, K_3$, $H(\{b_1, b_3\})$, $H(\{b_2, b_3\})$. We obtain a linear 2-covering $D'$ of the graph $H$ with a unique constituent containing the vertex $a$. This operation "corrected" the situation in the vertex $a$ (now $a$ belongs to exactly three constituents of the covering $C \cup D'$) and did not "worsen" the situation for any other vertex.

Analogously, for $VD_3 = \{b_1, b_2\}$, eliminate from $D$ the constituents $D_i, i = 1, 2, 3$, and add the new constituent $K$.

The truth of the implication (ii) $\Rightarrow$ (i) follows from Proposition 7 and Theorem B.

Theorem 3.2. For a graph $G$ with $\delta(G) \geq 5$ the following two statements are equivalent:

(i) $G \in \mathcal{L}_2$,

(ii) $G$ does not contain induced subgraphs isomorphic to the Beineke graphs $B_{11} - B_6$ (Fig. 1).

Proof. (ii) implies (i). Let $C = \{C_i : i = 1, \ldots, k\}$ be the list of maximal 2-large cliques of the graph $G$ and $V_i = V C_i$. Evidently, the wheel $W_6$ (Fig. 1, $B_1$) is the only graph of order 6 that has a dominating vertex and does not contain the induced star $K_{1,3}$ (Fig. 1, $B_2$) and $K_4$. Hence the following assertion is true.

Proposition 1. Every vertex of $G$ belongs to some clique $C_i$. 

Since \( G \) does not contain the induced subgraphs \( B_3 \) and \( B_4 \), the following two propositions follow immediately.

**Proposition 2.** If \( v \in VG \setminus V_i \), then \( v \) has at most two neighbors in \( V_i \).

**Proposition 3.** \( |V_i \cap V_j| \leq 1 \), \( i \neq j \).

**Proposition 4.** If \( v \in V_i \), then \( N_G[v] = V_i \) or \( N_G[v] = V_i \cup F \), where \( F \) is a clique such that \( F \cap V_i = \{v\} \).

Suppose to the contrary that it does not hold. By Proposition 2, \( G \) contains the induced star \( K_{1,3} \).

If the union of all cliques \( C_i \) coincides with \( G \), then \( C \) is a linear 2-covering of \( G \). Otherwise, remove from \( G \) the edges of all cliques \( C_i \) and any resulting isolated vertex. Denote the resulting graph by \( H \).

**Proposition 5.** \( H \) does not contain the path \( P_3 \) as an induced subgraph.

Suppose that \( VP_3 = \{a, b, c\} \) and \( a \) is the central vertex of \( P_3 \) in \( H \). By Proposition 3, \( G(a, b, c) = K_3 \). If \( bc \notin EH \), then \( bc \) is an edge of some clique \( C_1 \) from \( C \). The vertex \( a \) belongs to some other clique \( C_2 \) from \( C \) and, by Proposition 2, \( a \) is not adjacent to any vertex from \( C_1 \) differing from \( b \) and \( c \). Again by Proposition 2, since \( |C_2| \geq 4 \), \( C_2 \) contains the vertex \( f \) adjacent to neither \( b \) nor \( c \). In \( C_1 \) fix two new vertices \( d \) and \( e \). At least one of these vertices must be adjacent to \( f \), since otherwise \( G(a, b, c, d, e, f) = B_6 \). For instance, let \( f \) and \( d \) be adjacent. Then \( G(a, b, c, d, f) = B_6 \). Thus \( bc \in EH \).

By Proposition 5, each vertex of \( H \) belongs to exactly one maximal clique. Adding all such cliques to the list \( C \), we obtain a linear 2-covering of the graph \( G \).

**Theorem 3.3.** For \( r > 3 \) and an arbitrary constant \( c \), the set of all graphs \( G \) in \( \mathcal{L}_r^t \) with \( \delta(G) \geq c \) cannot be characterized by a finite list of forbidden induced subgraphs.

**Proof.** Suppose to the contrary that \( S \) is such a list. In [7], the infinite family of graphs shown in Figure 3 is given. In the figure the graph \( K_4 - e \) is repeated \( t \) times, where \( t \) is arbitrary. The graph \( G_t \) has the following property: \( G_t \notin L_3 \), but \( G_t - v \notin L_3 \) for any vertex \( v \).

We “paste” \( r - 3 \) pairwise disjoint copies of \( K_s, s = \max\{r^2 - r + 2, c\} \), to each vertex of \( G_t \). Moreover, we require here that the \( K_s \) “pasted” to different vertices are pairwise disjoint also (Fig. 4). Denote the resulting graph by \( H_t \). Since \( G_t \notin L_3 \), it follows from Theorems A and B that \( H_t \notin L_3 \). But \( \delta(H_t) \geq c \), and hence \( H_t \) contains the induced subgraph \( F \) from the list \( S \). Further, if \( v \in VG_t \), then \( H_t - v \notin L_3 \), as \( G_t - v \notin L_3 \). Thus \( VG_t \subseteq VF_t \) for each \( t \). Since \( t \) is not restricted and the order of \( G_t \) grows with \( t \), the list \( S \) cannot be finite.

**FIGURE 3.** Graph \( G_t \).
2. ON THE ALGORITHM

Consider the recognition problem

\[ G \in L'_3 \] \quad K(G) \geq 19. \tag{4} \]

By Theorem B, it is necessary to construct a linear 3-covering \( E \) of the graph \( G \). The construction procedure for the next step adds to the resulting covering just one constituent, simultaneously removing its edges from the graph.

(1) First, construct a set \( C \subseteq E \) of maximal 3-large cliques of \( G \) covering \( V(G) \). Delete the edges of the cliques belonging to \( C \), denote the resulting graph by \( G_C \). If the graph satisfies (4), then such a set can be found.

(2) No four of the constructed cliques can have a common vertex, for otherwise we have a contradiction to Theorem A. If \( G_C \) is the empty graph, then we take the set \( C \) in the capacity of \( E \); we have \( G \in L'_3 \).

(3) Consider the subgraph \( V_C \subseteq V(G) \) of vertices belonging to exactly two cliques from \( C \). Let \( u \) be one of them. The induced subgraph \( G_u = G[(N_C(u))\setminus u] \) must be a clique. If \( N_C(u) \neq \emptyset \), then add \( G_u \) to \( C \) as a new constituent (deleting from \( G \) the corresponding edges). Do the same for each \( u \) from \( V_C \). Again denote by \( C \) the constructed part of the sought-for covering \( E \) and by \( G_C \) the remaining graph. If the new set \( C \) differs from the previous one, repeat steps (1)–(3).

(4) Each non-isolated vertex of the graph \( G_C \) belongs to exactly one clique from \( C \) and, by Theorem A, this clique must be a constituent of any linear 3-covering. Therefore, \( G \in L'_3 \) if and only if \( G_C \in L'_3 \). Delete all isolated vertices in \( G_C \). Then the sought-for covering \( E \) can be represented by the union \( E = C \cup D \), where \( D \) is an arbitrary linear 3-covering of the graph \( G_C \).

The solution of the recognition problem \[ G \in L'_3 \] is known (see, for example, [8]).

If one of the above conditions does not hold, then \( C \notin L'_3 \). Obviously, the described procedure has polynomial complexity (a polynomial in the vertex number).

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