PURE-PROJECTIVE MODULES

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Abstract. The question is addressed of when all pure-projective modules are direct sums of finitely presented modules. It is proved that this is the case over hereditary noetherian rings. Partial results are obtained for uniserial rings. Some of the methods are model-theoretic, and the techniques developed using these may be of interest in their own right.

1. Introduction

It is well known that a module is pure-projective if and only if it is a direct summand of a direct sum of finitely presented modules. Our principal concern is the question of when all pure-projective modules *are* direct sums of finitely presented modules (or, equivalently, of finitely generated modules, for finitely generated pure-projectives are finitely presented).[1] By a theorem of Maranda, the answer is *yes* for abelian groups, that is over the ring of integers, see [7, Thm. 30.2].

If the ring is Krull–Schmidt in the sense of [8], i. e., when every finitely presented module is a direct sum of modules with a local endomorphism ring, then the Krull–Remak–Schmidt–Azumaya theorem implies that, again, the answer is *yes*. All right noetherian, (two-sided) serial rings are examples of such rings (by Fact 5.1, which holds for serial rings as well, and [5, Prop. 9.24] or [12, Lemma 2.11]).

Our development of the decomposition theory of pure-projective modules can be outlined as follows.

After recollection of some model-theoretic preliminaries in Section 2, we introduce pure-projectives and develop some of their model theory in Section

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[1]This is a 'pure' version of the classical question of when projectives are free.

3—based on the second author's description of countably generated pure-projective modules, Fact 3.4. The main result in this section is what we call the telescoping map theorem, Theorem 3.7.

As a first application of this machinery, we give an easy proof of the Bass–Björk result, Theorem 4.2, that if every flat right R-module is projective, then every left R-module has the d.c.c. on finitely generated submodules.

In Section 5 we deal with uniserial rings. The decomposition theory of pure-projectives is little understood there. Even over a uniserial domain R it is possible to have an indecomposable pure-projective module which is not finitely generated. In [12, proof of Lemma 14.21] such a module was constructed in the category $\mathrm{Add}(M)$ of a *cyclically presented* module M, i. e. of a module of the form R/rR, where $r \in R$. Here $\mathrm{Add}(M)$ denotes the smallest additive category containing M (whose objects are the direct summands of direct sums of copies of M). We are able, however, to answer our original question within this restricted category: we completely determine when every (pure-projective) module in $\mathrm{Add}(R/rR)$ is a direct sum of finitely generated modules. This is the same as to determine when every projective module over the ring $S = \mathrm{End}(R/rR)$ is free, see Theorem 5.8.

In the final Section 6 we answer in the positive our original question for (two-sided) hereditary noetherian rings by showing that every pure-projective module over such a ring is a direct sum of finitely generated modules, Cor. 6.5. The heart of the proof is the special case of hereditary noetherian prime rings, Thm. 6.5. This section does not use the machinery developed earlier but is rather based on a general result, Lemma 6.3, saying that, over a semihereditary semiprime Goldie ring pure-projectives are torsion-splitting.

Unless stated otherwise, all our modules are right modules over an associative ring R with 1, that is, we work in the category Mod-R. All our unexplained terminology is standard and can be found in one of the texts cited. (In particular, as for ring-theoretic terms, when no side is specified, they are meant on either side; e. g., a hereditary noetherian ring is a two-sided hereditary, two-sided noetherian ring.)

2. Free realizations

Basic definitions from the model theory of modules can be found in [11]. Below we recall but a few.

As common, by a *tuple* we mean a finite sequence of elements, usually denoted by overbarred small letters. Slightly abusing notation, we write $\overline{m} \in M$ to mean that each entry of the tuple $\overline{m}$ is in M.

By a *pointed module* we mean a pair $(M, \overline{m})$ where M is a module and $\overline{m} \in M$ is a tuple of elements from M. A *morphism* of pointed modules (or a *pointed morphism*), $f : (M, \overline{m}) \to (N, \overline{n})$, is a homomorphism $f : M \to N$ such that $f(\overline{m}) = \overline{n}$ coordinatewise (in particular, $\overline{m}$ and $\overline{n}$ have to have the same length).

Recall that a *pp-formula* is a formula of the form $\exists\, \overline{y}\, (\overline{y}A = \overline{x}B)$, where $\overline{x}$, $\overline{y}$ are tuples of variables, and A, B are rectangular matrices over the ring of appropriate size. We often use the shorthand $A \mid \overline{x}B$ (read: 'A divides $\overline{x}B$') for this formula. For instance, $a \mid x$ stands for $\exists\, y\, (ya = x)$.

Given a pointed module $(M, \overline{m})$ and a pp-formula $\varphi(\overline{x})$ of the form $\exists\, \overline{y}\, (\overline{y}A = \overline{x}B)$, one writes $M \models \varphi(\overline{m})$ (and says, $\overline{m}$ satisfies φ in M) if there is a tuple $\overline{n} \in M$ such that $\overline{n}A = \overline{m}B$. For instance, $M \models (a \mid x)(m)$ iff $m \in Ma$.

We write $\varphi \to \psi$ if a tuple in a module satisfies ψ whenever it satisfies φ. (So this is a statement about the entire category of all right R-modules.)

The *pp-type* of $\overline{m}$ in M, denoted $pp_M(\overline{m})$, is the set of all pp-formulae satisfied by the tuple $\overline{m}$ in M.

It is easily seen that homomorphisms preserve pp-formulae: if $f : M \to N$ and $M \models \varphi(\overline{m})$, then $N \models \varphi(f(\overline{m}))$, hence also $pp_M(\overline{m}) \subseteq pp_N(f(\overline{m}))$. In particular, if $f : (M, \overline{m}) \to (N, \overline{n})$ is a pointed morphism and $M \models \varphi(\overline{m})$, then $N \models \varphi(\overline{n})$, hence also $pp_M(\overline{m}) \subseteq pp_N(\overline{n})$.

There are two well-known cases where the converse is true. (Another one will be dealt with in Lemma 3.3 below.)

Fact 2.1. [11, Prop. 8.5, Thm. 2.8] *Let M be finitely presented or N be pure-injective. Suppose that $\overline{m} \in M$ and $\overline{n} \in N$ are such that $pp_M(\overline{m}) \subseteq pp_N(\overline{n})$.*

Then there exists a pointed morphism $f : (M, \overline{m}) \to (N, \overline{n})$.

A pointed module $(M, \overline{m})$ is said to be a *free realization* of a pp-formula $\varphi(\overline{x})$ if 1) $M \models \varphi(\overline{m})$ and 2) if $N \models \varphi(\overline{n})$ for some tuple $\overline{n}$ in a module N, then there is a pointed morphism $(M, \overline{m}) \to (N, \overline{n})$. For instance, if a is an element of the ring R, then (R, a) is a free realization of the formula $a \mid x$, and $(R/aR, 1 + aR)$ is a free realization of $xa = 0$. Both of these are *finitely presented* in the sense that the underlying module is.

Free realizations are never unique: one can always add arbitrary modules or take pure injective envelopes without disturbing conditions 1) and 2) above. In general, there is no 'minimal' free realization either.

However, as the following shows, all pp-formulae have a free realization, even a 'small' one.

Fact 2.2. [11, Prop. 8.4] *Every pp-formula has a finitely presented free realization.*

The pp-type $p = pp_M(\overline{m})$ is called *finitely generated* if there exists a formula $\varphi \in p$ such that $\varphi \to \psi$ for every $\psi \in p$. In this case φ is said to *generate* p. It is easy to verify that for every free realization $(M, \overline{m})$ (of some pp-formula φ), the pp-type $pp_M(\overline{m})$ is finitely generated (by φ).

It is also not hard to prove that any pointed module $(N, \overline{n})$ with underlying finitely presented module N is a free realization of some pp-formula. This yields

Fact 2.3. [11, Prop. 8.4] *Every pp-type in a finitely presented module is finitely generated.*

In fact one can pinpoint the generating formula precisely. Let M be given by generators $\overline{n}$ and relations $\overline{n}A = 0$, and let $\overline{m} = \overline{n}B$. Then $pp_M(\overline{m})$ is generated by the formula $\exists\, \overline{y}\, (\overline{y}B = \overline{x} \wedge \overline{y}A = 0)$. This can be generalized as follows.

Fact 2.4. [16, Lemma 1.5] *Suppose $(M, \overline{m})$ is a pointed module such that $pp_M(\overline{m})$ is finitely generated, and $\overline{n}$ is contained in the submodule generated by $\overline{m}$. Then $pp_M(\overline{n})$ is finitely generated.*

More precisely, if $\overline{n} = \overline{m}A$ and $pp_M(\overline{m})$ is generated by $\varphi(\overline{x})$, then $pp_M(\overline{n})$ is generated by $\exists\, \overline{y}\, (\overline{y}A = \overline{x} \wedge \varphi(\overline{y}))$.

Proof. Let $\psi \in pp_M(\overline{n})$, and denote the formula $\psi(\overline{x}A)$ by $\delta(\overline{x})$. The latter formula is in $pp_M(\overline{m})$, hence $\varphi \to \delta$. To prove $\exists\, \overline{y}\, (\overline{y}A = \overline{x} \wedge \varphi(\overline{y})) \to \psi(\overline{x})$, let $\overline{a}$ satisfy $\exists\, \overline{y}\, (\overline{y}A = \overline{x} \wedge \varphi(\overline{y}))$ in some module N. We have to show that it also satisfies ψ. Choose witnesses $\overline{b}$ satisfying φ such that $\overline{b}A = \overline{a}$. As $\varphi \to \delta$, the tuple $\overline{b}$ also satisfies δ, i. e., $\overline{a} = \overline{b}A$ satisfies ψ, as desired. $\square$

We conclude with an algebraic description of finite generation of pp-types.

Lemma 2.5. *Let $(M, \overline{m})$ be a pointed module. Then the following are equivalent.*

(1) $p = pp_M(\overline{m})$ *is finitely generated.*

(2) *There is a finitely presented pointed module* $(N, \overline{n})$ *and a 'universal' pointed morphism* $f : (N, \overline{n}) \to (M, \overline{m})$*, i. e., given a finitely presented pointed module* $(K, \overline{k})$ *and a pointed morphism* $g : (K, \overline{k}) \to (M, \overline{m})$*, the diagram below can be completed as shown.*

$$\begin{tikzcd} & (K,\overline{k}) \arrow[dl, dashed, "h"'] \arrow[d, "g"] \\ (N,\overline{n}) \arrow[r, "f"] & (M,\overline{m}) \end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2). Let φ generate p. Pick a finitely presented free realization $(N, \overline{n})$ of φ and let f be the corresponding map.

We know that $pp_K(\overline{k})$ is generated by a certain $\psi(\overline{x})$. Applying g, we obtain $\psi \in p$, and therefore $\varphi \to \psi$, hence also $pp_K(\overline{k}) \subseteq pp_N(\overline{n})$. So the desired h exists by Fact 2.1.

(2) $\Rightarrow$ (1). Suppose that there is a universal pointed morphism $f : (N, \overline{n}) \to (M, \overline{m})$. Let φ generate $pp_N(\overline{n})$. We claim that φ generates p. Applying f, we obtain $\varphi \in p$.

Suppose that $\psi \in p$. Let $(K, \overline{k})$ be a free realization of ψ and $g : (K, \overline{k}) \to (M, \overline{m})$ the corresponding map. By the universal property, g factors through f via a pointed morphism $h : (K, \overline{k}) \to (N, \overline{n})$. Applying h to $K \models \psi(\overline{k})$, we obtain $N \models \psi(\overline{n})$. Thus $\varphi \to \psi$. $\square$

3. PURE-PROJECTIVE MODULES

A module is said to be *pure-projective* if it is projective with respect to pure-exact sequences. Warfield [20] showed that a module is pure-projective if and only if it is a direct summand of a direct sum of finitely presented modules, cf. [21, 33.6]. In other words, the class of pure-projective right R-modules is precisely the class of objects in Add(mod-R) (where, as usual, mod-R stands for the full subcategory of Mod-R of finitely presented right R-modules).

We start with a remark that is crucial when trying to find certain finitely presented direct summands in a pure-projective module.

Fact 3.1. [21, 34.1(3)] *If N is a finitely generated pure submodule of a pure-projective module M, then N is a direct summand of M. (For, by* [21, 34.1(2)], *the factor M/N is pure-projective, hence the pure-exact sequence $0 \to N \to M \to M/N \to 0$ splits.)*

Pure-projective modules share many properties with finitely presented modules, the most important model-theoretic one being the following, which was noticed by Ivo Herzog, cf. [16, remark before 2. 3 (a)].

Fact 3.2. *Every tuple in a pure-projective module has a finitely generated pp-type.*

Here is another similarity between pure-projective and finitely presented modules (cf. Fact 2.1 above).

Lemma 3.3. *Let M be a pure-projective and N an arbitrary module. Suppose that $\overline{m} \in M$ and $\overline{n} \in N$ are such that $pp_M(\overline{m}) \subseteq pp_N(\overline{n})$.*

Then there is a pointed morphism $f : (M, \overline{m}) \to (N, \overline{n})$. Consequently, $(M, \overline{m})$ is a free realization of any formula that generates $pp_M(\overline{m})$.

Proof. Being pure-projective, M is a direct summand of a direct sum of finitely presented modules M_i, $i \in I$. Decompose $\overline{m}$ as $(\overline{m}_1, \ldots, \overline{m}_k)$ in $\oplus_{i \in I} M_i$. If φ_i generates $pp_{M_i}(\overline{m}_i)$ then $\varphi = \varphi_1 + \ldots + \varphi_k$ generates $p = pp_M(\overline{m})$, hence, by hypothesis, $\varphi \in pp_N(\overline{n})$. Therefore we can write $\overline{n} = \overline{n}_1 + \ldots + \overline{n}_k$ so that $N \models \varphi_i(\overline{n}_i)$ for each i.

Since each M_i is finitely presented, there is a pointed morphism $f_i : (M_i, \overline{m}_i) \to (N, \overline{n}_i)$. Then, for $f = f_1 \oplus \ldots \oplus f_k : \oplus_{i \in I} M_i \to N$, we have $f(\overline{m}) = \overline{n}$. Now restrict f to M. $\square$

By a variant of Kaplansky's Theorem (see [5, Thm. 4.27]), every pure-projective module is a direct sum of countably generated (and automatically pure-projective) modules. This enables us to make extensive use of the following key fact, a partial converse to Fact 3.2 (which is implicit also in [17, Thm. 2.2]).

Fact 3.4. [16, Cor. 2.9 and Lemma 3.9] *A countably generated module is pure-projective if and only if all tuples have a finitely generated pp-type.*

We can improve on this by keeping track of the pp-formulae involved. Besides, it suffices to consider pp-types of generators.

Proposition 3.5. *Let $m_1, m_2, \ldots$ be a countable sequence of generators for a module M such that $\varphi_i(x_1, \ldots, x_i)$ generates $p_i = pp_M(m_1, \ldots, m_i)$. Then*

M is a direct summand of $N = \oplus_{i\geq 1} N_i$, where N_i is a finitely presented free realization of $\varphi_i(0, \dots, 0, x)$. In particular, M is pure-projective.

Proof. Let (N_i, n_i') be a finitely presented free realization of $\varphi_i(0, \dots, 0, x)$, and set $M_i = \oplus_{k=1}^{i} N_k$. By induction on i we will construct elements $n_i \in M_i \subseteq N$ and an ascending chain of morphisms $f_i : M_i \to M$ such that $M_i \models \varphi_i(n_1, \dots, n_i)$ and $f_i(n_j) = m_j$ for every $j \leq i$. Then the map $f = \bigcup_{i\geq 1} f_i$ is an epimorphism $N \to M$.

Further, the choice of φ_i yields the type inclusion

$$pp_M(m_1, \dots, m_i) \subseteq pp_{M_i}(n_1, \dots, n_i) = pp_N(n_1, \dots, n_i).$$

Therefore $g(m_i) = n_i$ defines a morphism $M \to N$, for if $m_1 r_1 + \ldots + m_i r_i = 0$, by this inclusion, also $n_1 r_1 + \ldots + n_i r_i = 0$. Since $fg(m_i) = f(n_i) = m_i$ for every i, the map f splits, and so M is a direct summand of N, as desired.

It remains to find n_i as described. For $i = 1$ take $n_1 = n_1'$. Since N_1 is finitely presented, there is a pointed morphism $f_1 : (N_1, n_1) \to (M, m_1)$.

Suppose we have already constructed n_k and f_k for every $k \leq i$. Clearly $\varphi_i(x_1, \dots, x_i) \to \exists x_{i+1}\, \varphi_{i+1}(x_1, \dots, x_i, x_{i+1})$. Since $M_i \models \varphi_i(n_1, \dots, n_i)$, there is $n_{i+1}'' \in M_i$ such that $M_i \models \varphi_{i+1}(n_1, \dots, n_i, n_{i+1}'')$. Applying f_i we obtain $M \models \varphi_{i+1}(m_1, \dots, m_i, f_i(n_{i+1}''))$. But also $M \models \varphi_{i+1}(m_1, \dots, m_i, m_{i+1})$, and so $M \models \varphi_{i+1}(0, \dots, 0, m_{i+1} - f_i(n_{i+1}''))$. Since (N_{i+1}, n_{i+1}') is a free realization of $\varphi_{i+1}(0, \dots, 0, x)$, there is a pointed morphism $f_{i+1}' : (N_{i+1}, n_{i+1}') \to (M, m_{i+1} - f_i(n_{i+1}''))$. Now take $f_{i+1} = f_i \oplus f_{i+1}'$ and $n_{i+1} = n_{i+1}'' + n_{i+1}'$. Then

$$f_{i+1}(n_{i+1}) = f_i(n_{i+1}'') + f_{i+1}'(n_{i+1}') = f_i(n_{i+1}'') + m_{i+1} - f_i(n_{i+1}'') = m_{i+1}\,.$$

Finally, adding $M_i \models \varphi_{i+1}(n_1, \dots, n_i, n_{i+1}'')$ and $N_{i+1} \models \varphi_{i+1}(0, \dots, 0, n_{i+1}')$ we obtain $M_{i+1} \models \varphi_{i+1}(n_1, \dots, n_i, n_{i+1})$. □

In the following lemma, which will be used in the next section, we address the question about when a pp-type in a direct limit is finitely generated.

Lemma 3.6. *Let $\langle (M_i, \overline{m}_i), f_{ij} \rangle$ be a directed system of pointed modules over the directed set I, where $(M_i, \overline{m}_i)$ is a free realization of φ_i for every $i \in I$. Let further $\langle (M, \overline{m}), f_i \rangle$ be the corresponding direct limit. Then the following are equivalent.*

(1) *$pp_M(\overline{m})$ is finitely generated;*
(2) *there exists i such that $\varphi_i \to \varphi_j$ for every $j \geq i$.*

(3) *there exists i such that all φ_j with $j \geq i$ are equivalent to φ_i.*

Whenever these conditions hold, $pp_M(\overline{m})$ is generated by φ_i.

Proof. Applying f_{ij} to $M_i \models \varphi_i(\overline{m}_i)$ we obtain $M_j \models \varphi_i(\overline{m}_j)$, hence $\varphi_j \to \varphi_i$ for all $j \geq i$. Similarly, $f_i : M_i \to M$ yields $\varphi_i \in pp_M(\overline{m})$ for every i. Denote this latter type by p.

(1) $\Rightarrow$ (3). Assume, p is generated by φ. As $\varphi_j \in p$, it follows that $\varphi \to \varphi_j$ for every j. But $M \models \varphi(\overline{m})$ implies $M_i \models \varphi(\overline{m}_i)$ for sufficiently large i. Since $(M_i, \overline{m}_i)$ is a free realization of φ_i, we obtain $\varphi_i \to \varphi$ for these i, hence also $\varphi_i \to \varphi \to \varphi_j \to \varphi_i$ for every $j \geq i$.

(2) $\Rightarrow$ (1). Suppose i is as in (2). We claim φ_i generates p. We know already that $\varphi_i \in p$. If now $\psi \in p$, as above it follows that $\varphi_j \to \psi$ for sufficiently large j. Choosing $j \geq i$, by (2), we obtain $\varphi_i \to \varphi_j \to \psi$.

As (3) $\Rightarrow$ (2) is trivial, this completes the proof. □

The next result, which was inspired by Dung and Facchini [3, Thm. 4.9] (see also [5, Prop. 9.30]), will be used in the main theorem on uniserial rings below. Some more motivation may be in order. Although this is not the point of view taken in the next proof, a direct limit (M, f_i) of a direct system of modules, (N_i, f_{ij}), $i, j \in I$, can be regarded as an epimorphic image of the direct sum $\oplus_{i \in I} N_i$. Further, the corresponding epimorphism is easily seen to be pure, cf. [21, 33.9 (2)]. So, if M itself is pure-projective, this map splits, i. e., M is a direct summand of $\oplus_{i \in I} N_i$. However, this is a mere existence statement, which does not indicate the complement, let alone the map constituting the splitting. This exactly is what the (proof of the) next theorem does—in the case that M is countably generated. (Note that it also constitutes a generalization of Eilenberg's trick.)

Theorem 3.7 (The telescoping map)**.** *Suppose that M is a countably generated pure-projective module. If M is the direct limit of a direct system (N_i, f_{ij}), $i, j \in I$, then $M \oplus \oplus_{i \in I} N_i \cong \oplus_{i \in I} N_i$.*

Proof. Represent the elements of M as (equivalence classes of) pairs (m, i) with $i \in I$ and $m \in N_i$, and where $(m, i) = (n, j)$ if there exists $k \geq i, j$ such that $f_{ik}(m) = f_{jk}(n)$.

Let $m_1, m_2, \ldots$ be a countable list of generators of M. By Fact 3.4, the pp-type of any finite tuple of elements from this list is generated by a single formula. Therefore there is a chain of indices $i_1 < i_2 < \ldots \in I$ such that

$m_k = (n_k, i_k)$ and the tuple $(m_1, \ldots, m_{k-1}, m_k)$ has the same pp-type in M as the tuple $(f_{i_1,i_k}(n_1), \ldots, f_{i_{k-1},i_k}(n_{k-1}), n_k)$ in N_{i_k}.

Denote $\{i_1, i_2, \ldots\}$ by I'. First we show that $M \cong \varinjlim_{i,j\in I'}(N_i, f_{ij})$.

Since any element $m \in M$ is a linear combination of some of the (n_k, i_k), we may write m as (n, j), where j is the highest occurring index from I' (in that linear combination) and n is the corresponding linear combination of the $f_{i_k,j}(n_k)$. Hence, the natural map $\varinjlim_{i,j\in I'}(N_i, f_{ij}) \to \varinjlim_{i,j\in I}(N_i, f_{ij})$ (induced by the inclusion $I' \subseteq I$) is onto.

To prove that this map is also mono, let $m = (n, j)$ go to zero, i.e., $m = 0$ in M. Notice that the aforementioned equality of pp-types is preserved by passing to the corresponding linear combinations, hence $pp_M(m) = pp_{N_j}(n)$, and so $n = 0$ as well. But then $m = (n, j) = 0$ also in $\varinjlim_{i,j\in I'}(N_i, f_{ij})$.

Thus we may assume that $M = \varinjlim_{i,j,\in I'}(N_i, f_{ij})$, and if we prove $M \oplus \oplus_{i\in I'}N_i \cong \oplus_{i\in I'}N_i$, then adding $\oplus_{i\in I\setminus I'}N_i$ to both sides finishes off the proof. We may therefore even assume that $I = \omega = \{1, 2, \ldots\}$, that $m_i = (n_i, i)$ with $n_i \in N_i$, and that $pp_M(m_1, \ldots, m_{k-1}, m_k) = pp_{N_k}(f_{1k}(n_1), \ldots, f_{k-1,k}(n_{k-1}), n_k)$. Then every element $m = m_1r_1+m_2r_2+\ldots+m_kr_k$ in M has a representation (n, k) with $n = f_{1,k}(n_1)r_1 + f_{2,k}(n_2)r_2 + \ldots + n_kr_k$.

Since M is pure-projective, by Lemma 3.3, there exists a morphism $g_k : M \to N_k$ such that $g_k(m_i) = f_{ik}(n_i)$ for every $i \leq k$. It is easily checked that therefore $g_k(m) = n$ when $m = (n, k)$ is a representation as in the previous paragraph.

Let $N = \oplus_{i\in I}N_i$ and define $g : M \to N$ by sending $m \in M$ to

$$(g_1(m), g_2(m) - f_{12}g_1(m), \ldots, g_k(m) - f_{k-1,k}g_{k-1}(m), \ldots).$$

Since $g_{j+1}(m_i) = f_{j,j+1}g_j(m_i)$ for every $j \geq i$, this map is well defined.

We claim, $\ker(g) = 0$ (and so $\mathrm{im}(g) \cong M$). Indeed, let $m \in M$ be in the kernel and represent it by $m = (n, k)$ as above so that $g_k(m) = n$. From $g(m) = 0$ we successively obtain $0 = g_1(m) = \ldots = g_k(m) = n$, hence $m = (0, k)$ is zero in M.

Now define $h : N \to M$ by $h(l_1, \ldots, l_k, 0, \ldots) = (l_1, 1)+\ldots+(l_k, k)$. Then

$$hg(m_i) = h(g_1(m_i), g_2(m_i) - f_{12}g_1(m_i), \ldots, g_i(m_i) - f_{i-1,i}g_{i-1}(m_i), 0, \ldots) =$$

$$(g_1(m_i), 1)+(g_2(m_i), 2)-(f_{12}g_1(m_i), 2)+\ldots+(g_i(m_i), i)-(f_{i-1,i}g_{i-1}(m_i), i) =$$

$$= (g_i(m_i), i) = (n_i, i) = m_i.$$

Thus g splits, i.e., $N = \mathrm{im}(g) \oplus \ker(h)$. As $\mathrm{im}(g) \cong M$, it remains to prove that $\ker(h) \cong N$. To this end, let $s : N \to N$ be the map that sends $n = (l_1, l_2, \ldots, l_k, 0, \ldots)$ to $(l_1, l_2 - f_{12}(l_1), \ldots, l_k - f_{k-1,k}(l_{k-1}), -f_{k,k+1}(l_k), 0, \ldots)$.

We claim, $\ker(s) = 0$; indeed, if $s(n) = 0$, we successively obtain $0 = l_1 = \ldots = l_k$, whence $n = 0$. Hence $\mathrm{im}(s) \cong N$, and so it suffices to prove $\mathrm{im}(s) = \ker(h)$.

For $\mathrm{im}(s) \subseteq \ker(h)$, consider $n = (l_1, l_2, \ldots, l_k, 0, \ldots) \in N$. Then $hs(n) = (l_1, 1) + (l_2, 2) - (f_{12}(l_1), 2) + \ldots + (l_k, k) - (f_{k-1,k}(l_{k-1}), k) - (f_{k,k+1}(l_k), k+1) = 0$, since $(l_i, i) = (f_{i,i+1}(l_i), i+1)$ for all i.

We are left with $\ker(h) \subseteq \mathrm{im}(s)$. Let n be as before and $h(n) = 0$, i.e., $(l_1, 1) + \ldots + (l_k, k) = 0$. By the definition of direct limit, there is $j \geq k$ such that $f_{1j}(l_1) + \ldots + f_{kj}(l_k) = 0$. We may take j to be strictly bigger than k.

Consider the element $n' \in N$ whose ith component n'_i is defined as follows. If $i \leq k$, set $n'_i = f_{1i}(l_1) + \ldots + f_{i-1,i}(l_{i-1}) + l_i$. If $k < i \leq j$, set $n'_i = f_{k,i}(n'_k)$. For $i > j$, set $n'_i = 0$. Then easy calculations show that $n'_{i+1} - f_{i,i+1}(n'_i) = l_{i+1}$ for all $i < k$ and $n'_{i+1} - f_{i,i+1}(n'_i) = 0$ for $k \leq i < j$, while $n'_{j+1} - f_{j,j+1}(n'_j) = -f_{j,j+1}(n'_j) = 0$ because $n'_j = f_{1j}(l_1) + \ldots + f_{kj}(l_k) = 0$ by the choice of j.

Consequently, $s(n') = (l_1, l_2, \ldots, l_k, 0, \ldots) = n$, as desired, which completes the proof. □

In the proof of Theorem 5.8 we need one more technical fact about pure-projectives. For this, recall that a module is called *Bezout* if each of its finitely generated submodules is cyclic. A module M is said to be *endo-Bezout* if ${}_S M$ is Bezout, where $S = \mathrm{End}(M)$.

Lemma 3.8. *Given a pure-projective module M, the following are equivalent.*

(1) *M is endo-Bezout.*
(2) *For any one-place pp-formulae φ and ψ, if the pp-type of some element of M is generated by φ and the pp-type of some element of M is generated by ψ, then there is an element of M whose pp-type is generated by $\varphi + \psi$.*

Proof. Let m and n be arbitrary elements of M such that $pp_M(m)$ is generated by φ and $pp_M(n)$ is generated by ψ.

(1) $\Rightarrow$ (2). Condition (1) yields an element $k \in M$ such that $Sm + Sn = Sk$. We claim, $r = pp_M(k)$ is generated by $\varphi + \psi$.

Write $k = fm + gn$, where f and g are in S. Then $M \models \varphi(m)$ implies $M \models \varphi(fm)$, and $M \models \psi(n)$ implies $M \models \psi(gn)$. Then $M \models (\varphi + \psi)(k)$, i. e., $\varphi + \psi \in r$.

It remains to prove that $\varphi + \psi \to \xi$ for every $\xi \in r$. Indeed, from $m \in Sk$ and $M \models \xi(k)$ we obtain $M \models \xi(m)$, hence $\varphi \to \xi$, as $pp_M(m)$ is generated by φ. Similarly $\psi \to \xi$, and so $\varphi + \psi \to \xi$.

(2) $\Rightarrow$ (1). Condition (2) yields an element $k \in M$ whose pp-type is generated by $\varphi + \psi$. As $\varphi \to \varphi + \psi$, Lemma 2.1 implies $m \in Sk$. Similarly, $n \in Sk$, hence $Sm + Sn \subseteq Sk$.

If, conversely, $l \in Sk$, we have $M \models (\varphi + \psi)(l)$. Write $l = m' + n'$ accordingly so that $M \models \varphi(m') \wedge \psi(n')$. As above there are $f, g \in S$ such that $f(m) = m'$ and $g(n) = n'$. Then $l = m' + n' \in Sm + Sn$. $\square$

4. A result of Bass and Björk revisited

As a first application of our techniques, using an idea of Sakhayev [18] we give a new proof of a result of Bass and Björk, Theorem 4.2 below—a proof that does not use Björk's Theorem. We start with a special case of Prest's Lemma.

Remark 4.1. *Let φ be the pp-formula $A \mid \overline{x}$ and ψ the pp-formula $B \mid \overline{x}$. Then $\varphi \to \psi$ iff there is C such that $A = CB$.*

Proof. If $A = CB$, then the implication $\varphi \to \psi$ is obvious. For the converse, assume $\varphi \to \psi$, and let M be a free module generated by a tuple $\overline{m}$ of the same length as $\overline{x}$. Since $\overline{m}A$ trivially satisfies φ, it also satisfies ψ, i. e., $M \models B \mid \overline{m}A$. Therefore there is some matrix C such that $(\overline{m}C)B = \overline{m}A$, hence $A = CB$, as desired. $\square$

We emphasize the fact that the following new proof makes no reference to Björk's Theorem.

Theorem 4.2 (Bass–Björk)**.** *If every flat right R-module is projective, then every left R-module has the d.c.c. on finitely generated submodules.*

Proof. Let $M_1 \supseteq M_2 \supseteq \ldots$ be a chain of left R-modules with each M_i generated by some $\overline{m}_i$. Then $\overline{m}_{i+1} = B_i \overline{m}_i$ for appropriate matrices B_i.

Correspondingly, let N be the right module with generators $\overline{n}_1, \overline{n}_2, \dots$ and relations $\overline{n}_i = \overline{n}_{i+1} B_i$, $i = 1, 2, \dots$.

$$\begin{tikzcd} \vdots \\ \overline{n}_3 \arrow[d, "B_2"] \\ \overline{n}_2 \arrow[d, "B_1"] \\ \overline{n}_1 \end{tikzcd}$$

Consider the free module N_i generated by a tuple $\overline{x}_i$ (matching $\overline{n}_i$) and the morphism $f_{i,i+1} : N_i \to N_{i+1}$ given by $\overline{x}_i \mapsto \overline{x}_{i+1} B_i$. It is easy to verify that N, together with the maps $f_i : N_i \to N$ given by $f_i(\overline{x}_i) = \overline{n}_i$, is the direct limit of the thus given directed system.

By Fact 2.4, the pp-type of $f_{ij}(\overline{x}_i)$ in N_j is generated by the formula $B_{j-1} \cdot \ldots \cdot B_i \mid \overline{x}$. In particular, the pp-type of $f_{1j}(\overline{x}_1)$ in N_j is generated by the formula $B_{j-1} \cdot \ldots \cdot B_1 \mid \overline{x}$, which we denote by φ_j. It is easy to see that, in fact, $(N_j, f_{1j}(\overline{x}_1))$ is a free realization of φ_j.

Being a direct limit of free modules, N is flat, hence, by hypothesis, projective. In particular, N is pure-projective, and so, by Fact 3.2, $pp_N(\overline{n}_1)$ is finitely generated. Therefore Lemma 3.6 yields i such that $\varphi_i \to \varphi_j$ for every $j \geq i$, i. e., $B_{i-1} \cdot \ldots \cdot B_1 \mid \overline{x} \to B_{j-1} \cdot \ldots \cdot B_1 \mid \overline{x}$. By Remark 4.1, there is a matrix C_{j-1} such that $C_{j-1} B_{j-1} \cdot \ldots \cdot B_1 = B_{i-1} \cdot \ldots \cdot B_1$. But then $\overline{m}_i = B_{i-1} \cdot \ldots \cdot B_1 \overline{m}_1 = C_{j-1} B_{j-1} \cdot \ldots \cdot B_1 \overline{m}_1 = C_{j-1} \overline{m}_j$ for every $j \geq i$. Consequently, the chain $M_1 \supseteq M_2 \supseteq \ldots$ stabilizes after i steps. $\square$

Notice, applying Theorem 3.7 to the directed system leading to N above, we obtain $N \oplus R^{(\omega)} \cong R^{(\omega)}$, a fact that is known as Eilenberg's trick for *any* (countably generated) projective module.

5. Uniserial rings

The main objective of this section is to study the category $\mathrm{Add}(M)$ for a cyclically presented module $M = R/rR$ over a uniserial ring R. (Recall that a module M is *uniserial* if the lattice of submodules of M is a chain, while a ring R is *uniserial* if R_R and ${}_RR$ are uniserial modules.) We will determine when exactly this category is *trivial* in the sense that each module in $\mathrm{Add}(M)$ is a direct sum of copies of M. (Recall, the objects of $\mathrm{Add}(M)$ are the direct summands of all the direct sums $M^{(I)}$ of copies of M, so all of them are pure-projective.) In other words, we will answer our principal question stated in the introduction for this very special kind of pure-projective. For the remainder of this section, we will work in this setting of $\mathrm{Add}(M)$ where M is a fixed cyclically presented module R/rR over a uniserial ring R, where without loss $r \in \mathrm{Jac}(R)$. For, every uniserial ring is local; hence, if r is not in the Jacobson radical, then r is a unit and so $R/rR = 0$. Note that (even for an arbitrary finitely generated right module M) the study of $\mathrm{Add}(M)$ is equivalent to the study of projective right modules over $S = \mathrm{End}(M)$, since there exists an equivalence between $\mathrm{Add}(M)$ and the category of projective right S-modules, see [1, Lemma 29.4] or [5, Thm. 4.7].

Any cyclically presented R-module is uniserial and hence indecomposable. The next fundamental result states that these are the building blocks for the finitely presented modules.

Fact 5.1. (Drozd and Warfield, see [12, Thm. 2.3]) *Every finitely presented module over a uniserial ring can be decomposed into a finite direct sum of uniserial cyclically presented modules.*[2]

It was noticed by the first author ([12, Prop. 2.20], see also [5, Thm. 9.19]) that such a decomposition is unique.

The following result sheds some light on the structure of the category $\mathrm{Add}(M)$.

Fact 5.2. [3, Cor. 2.8] *If S is the endomorphism ring of a uniserial module M, then every finitely generated projective right S-module is free. Consequently, every finitely generated module in* $\mathrm{Add}(M)$ *is isomorphic to a direct sum of finitely many copies of M.*

As noted in [5, Prop. 3.12 (see also the remarks thereafter)], homing into the ring constitutes a duality between the finitely generated projective left modules and the finitely generated projective right modules (even for an arbitrary ring), and therefore all the left ones are free if and only if so are all the right ones. Thus, for S, the endomorphism ring of a uniserial module, both the left *and* the right finitely generated projectives are free. Note that such rings are even *projective free* in the terminology of Cohn [2, §0.2] (another left–right symmetric notion, see [2, remark after Prop. 0.2.7]), which means that finitely generated projectives are free *of unique rank*. Indeed,

[2]This holds also for serial rings, i. e. rings R such that both $_RR$ and R_R are direct sums of uniserial modules.

being semilocal, S has finite dual Goldie dimension. Therefore the rank of every finitely generated free right module is uniquely determined.

It is known that there are uniserial domains over which $\mathrm{Add}(M)$ can have a nontrivial decomposition theory, i. e., there exists a projective right S-module that is not free, see [13, Cor. 8.2]. We will give a precise criterion for this to happen.

First we establish some important properties of cyclically presented modules over uniserial rings.

Lemma 5.3. *Every cyclically presented module over a uniserial ring is endo-Bezout.*

Proof. According to [12, Cor. 11.15], every finitely presented module over a uniserial ring is endo-distributive (that is distributive as a module over its own endomorphism ring). Since M is uniserial, by [5, Thm. 9.1], $S = \mathrm{End}(M)$ has at most two maximal right ideals, and $S/\mathrm{Jac}(S)$ is a direct sum of at most two skew fields. Now, by [19, 3.33], distributivity is equivalent to the Bezout property over any ring which is abelian (von Neumann) regular after factorization by the Jacobson radical. $\square$

Note that every pure-projective module over a uniserial ring is endo-distributive, but it is not known if it has to be endo-Bezout.

The previous lemma allows us to describe generation of pp-types in (pure-projective) modules in $\mathrm{Add}(M)$.

Lemma 5.4. *Let R be a uniserial ring and $M = R/rR$, where $0 \neq r \in \mathrm{Jac}(R)$.*

The pp-types of non-zero elements in any $N \in \mathrm{Add}(M)$ are generated by formulae of the form $a \mid x \wedge xb = 0$, where $0 \neq a \in R$ and $0 \neq b \in \mathrm{Jac}(R)$ such that $r = ab$.

Further, for non-zero elements $m \in M$, the ring element a can be chosen in the rR-coset m itself.

Proof. We first prove the assertion for M. So let $m = a + rR$ be a non-zero element of M. As $(R/rR, 1+rR)$ is a free realization of $xr = 0$, Fact 2.4 tells us that (M, m) is a free realization of the formula $\exists\, y\,(ya = x \wedge yr = 0)$. As R is uniserial, either $a \in rR$, which contradicts $m \neq 0$, or else $r = ab$ for some $b \in R$. Then, as is easily seen, b must be in $\mathrm{Jac}(R)$ and $\exists\, y\,(ya = x \wedge yr = 0)$ is equivalent to $a \mid x \wedge xb = 0$.

In order to prove this for any $N \in \mathrm{Add}(M)$, let $0 \neq n \in N$. Then N is pure-projective, and $pp_N(n)$ is generated by a formula $\varphi_1 + \ldots + \varphi_k$, where φ_i generates the pp-type of some $m_i \in M$. As M is endo-Bezout, Lemma 3.8 yields some element $0 \neq m \in M$ with $pp_N(n) = pp_M(m)$, and the assertion follows from the first part of the proof. $\square$

Next we cite two more technical facts, where, for $r, s \in R$, we write $r \leq s$ if $RrR \supseteq RsR$, and $r \sim s$ if $RrR = RsR$.

Fact 5.5. [14, Cor. 3.2] *Let R be a uniserial ring and $0 \neq a, b, c \in \mathrm{Jac}(R)$. Then the following are equivalent.*

(1) $a \mid x \wedge xb = 0 \to ca \mid x$.
(2) $a \sim cab$.

Notice, $a \sim cab$ is equivalent to $a \in RcabR$, for $a \leq cab$ always holds.

Fact 5.6. [12, Prop. 2.21] *Let r, s be elements of a uniserial ring R. Then the following are equivalent.*

(1) $R/rR \cong R/sR$.
(2) $r \sim s$ *(i. e., $RrR = RsR$).*
(3) $r = usv$ *for some units $u, v \in R$.*

The following construction is used in the theorem below and may be of interest in its own right.

Lemma 5.7. *Suppose $r_1, r_2, \ldots \in \mathrm{Jac}(R)$ are such that $r_i \cdot \ldots \cdot r_2 \neq 0$ for every $i > 1$.*

Then the module N with generators $n_1, n_2, \ldots$ and relations $n_1 r_1 = 0$ and $n_{i+1} r_{i+1} = n_i$ for all $i \geq 1$, is not finitely generated..

Proof. Depict N as shown.

$$\begin{tikzcd} \vdots \\ n_2 \arrow[d, "r_2"] \\ n_1 \arrow[d, "r_1"] \\ 0 \end{tikzcd}$$

If N were finitely generated, it would be generated by a single n_k. We show this is impossible. In fact, we will reach a contradiction from the assumption that $n_{k+1} = n_k s$ for some $s \in R$.

Indeed, as $n_k = n_{k+1}r_{k+1}$, this assumption yields $n_{k+1}(1 - r_{k+1}s) = 0$, hence $n_{k+1} = 0$, for, by hypothesis, $1 - r_{k+1}s$ is a unit. Then $n_{k+1} = n_k = \ldots = n_1 = 0$. By the definition of N, n_1 must be a linear combination of the "relations," whence

$$n_1 = n_1 r_1 t_1 + (n_2 r_2 - n_1)t_2 + \ldots + (n_m r_m - n_{m-1})t_m$$

for some ring elements t_i. Comparing coefficients we obtain

$$1 = r_1 t_1 - t_2, \quad 0 = r_2 t_2 - t_3, \quad \ldots, \quad 0 = r_{m-1}t_{m-1} - t_m, \quad 0 = r_m t_m.$$

These imply

$$0 = r_m t_m = r_m r_{m-1} t_{m-1} = \ldots = r_m \cdot \ldots \cdot r_2 t_2 = r_m \cdot \ldots \cdot r_2(r_1 t_1 - 1).$$

The right hand factor is a unit again, so $r_m \cdot \ldots \cdot r_2$ must be 0, contradicting the hypothesis. □

We are ready now to give the promised criterion.

Theorem 5.8. *Let R be a uniserial ring, $0 \neq r \in \mathrm{Jac}(R)$, $M = R/rR$ and $S = \mathrm{End}(M)$. Then the following are equivalent.*

(1) $\mathrm{Add}(M)$ *is trivial, i. e., every module in* $\mathrm{Add}(M)$ *is isomorphic to some* $M^{(I)}$.
(2) *Every projective right S-module is free.*
(3) *If $r \sim a \sim b$ in R, then $r \neq ab$;*
(4) *Every non-zero element of any module $N \in \mathrm{Add}(M)$ is contained in a direct summand of N isomorphic to M.*
(5) *Every module in* $\mathrm{Add}(M)$ *contains a direct summand isomorphic to M;*
(6) *Every module in* $\mathrm{Add}(M)$ *contains a non-zero finitely generated direct summand.*
(7) *Every module in* $\mathrm{Add}(M)$ *is a direct sum of finitely generated modules.*

Proof. The equivalence of (1) and (2) follows from the aforementioned equivalence of categories. The equivalence of (5) and (6) as well as that of (1) and (7) follow from Fact 5.2. The implication (5) $\Rightarrow$ (1) is [4, Cor. 2.10]. Since (4) $\Rightarrow$ (5) is trivial, we are left with (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4).

(1) $\Rightarrow$ (3). Assume, on the contrary, that there are $a, b \in R$ such that $ab = r$ and $a, b \sim r$. In particular $0 \neq a, b \in \mathrm{Jac}(R)$. We will produce a module N in $\mathrm{Add}(M)$ which is not of the form $M^{(I)}$.

To this end we choose as follows non-zero elements $r_1, r_2, \dots$ in $\mathrm{Jac}(R)$ such that $r_i \sim r$ and $r_i \cdot \ldots \cdot r_1 \sim r$ for every i.

Since $a \sim b \sim ab = r$, we may start with $r_1 = b$ and $r_2 = a$. Now suppose, $r_1, \dots, r_i$ have already been chosen so that $r_i \cdot \ldots \cdot r_1 \sim r$. Then $r_i \cdot \ldots \cdot r_1 \sim b$. By Fact 5.6, there are units $u, v \in R$ such that $r_i \cdot \ldots \cdot r_1 = ubv$. Then $u^{-1} r_i \cdot \ldots \cdot r_1 = bv$, hence $au^{-1} r_i \cdot \ldots \cdot r_1 = abv = rv \sim r$, whence we may take $r_{i+1} = au^{-1}$.

We claim that the module N with generators $n_1, n_2, \dots$ and relations $n_1 r_1 = 0$ and $n_{i+1} r_{i+1} = n_i$ for $i \geq 1$ is a uniserial pure-projective that is (countably but) not finitely generated.

As cyclic modules over uniserial rings are uniserial and N is the union of a chain of cyclic modules, we see that N is uniserial. Further, Lemma 5.7 says that N is not finitely generated. But it is countably generated, and so to show that it is pure-projective, by Proposition 3.5, it suffices to check that every $p_i = pp_N(n_i)$ is finitely generated, which we do next.

Fix i, and consider, for all $j > i$, the submodule $N_j = n_j R$ of N. As noted before Fact 5.5, $pp_{N_j}(n_i)$ is generated by the formula $r_j \cdot \ldots \cdot r_{i+1} \mid x \wedge x r_i \cdot \ldots \cdot r_1 = 0$, which we denote by φ_j. Given $k > j > i$, we clearly have $\varphi_k \to \varphi_j$. We claim these formulae are in fact equivalent, for which we just have to show that $\varphi_j \to r_k \cdot \ldots \cdot r_{i+1} \mid x$. But this latter follows from Fact 5.5 with $c = r_k \cdot \ldots \cdot r_{j+1}$, $a = r_j \cdot \ldots \cdot r_{i+1}$, and $b = r_i \cdot \ldots \cdot r_1$, for $r \sim r_j \leq r_j \cdot \ldots \cdot r_{i+1} = a \leq cab = r_k \cdot \ldots \cdot r_1 \sim r$.

Thus the formulae φ_j are equivalent for all $j > i$. Therefore, the types $pp_{N_j}(n_i)$ are the same for all $j > i$ as well. Hence they are the same as $pp_N(n_i)$, which means that the latter is finitely generated, as desired.

Now that we have proved the claim about N, we may apply to it the (telescoping) Theorem 3.7 to obtain the decomposition

$$N \oplus \oplus_i R/(r_i \cdot \ldots \cdot r_1 R) \cong \oplus_i R/(r_i \cdot \ldots \cdot r_1 R).$$

But $R/(r_i \cdot \ldots \cdot r_1 R) \cong R/rR$ by Fact 5.6. Consequently, $N \oplus M^{(\omega)} \cong M^{(\omega)}$, whence $N \in \mathrm{Add}(M)$.

Finally, if N were of the form $M^{(I)}$, being uniserial it would have to be isomorphic to M, which is impossible, as the latter is finitely generated and N is not.

(3) $\Rightarrow$ (4). Suppose N is in $\mathrm{Add}(M)$ and $0 \neq n \in N$. We must prove that n is contained in a direct summand isomorphic to M. By Lemma 5.4, $pp_N(n)$ is generated by a formula of the form $a \mid x \wedge xb = 0$, where $ab = r$, $a \neq 0$, and $0 \neq b \in \mathrm{Jac}(R)$.

If a is invertible, $pp_N(n)$ is obviously generated by the quantifier-free formula $xa^{-1}r = 0$, and hence nR is a (finitely generated) pure submodule of the pure-projective module N, and consequently, by Remark 3.1, a direct summand. It remains to notice that $nR \cong R/a^{-1}rR \cong R/rR \cong M$ to see that in this case we are done.

So assume, a is not invertible. Then $0 \neq a, b \in \mathrm{Jac}(R)$, which will allow us to apply Fact 5.5 as follows.

Choose $n_1 \in N$ such that $n_1 a = n$. The type $pp_N(n_1)$, too, is generated by a formula $a_1 \mid x \wedge xb_1 = 0$ for some $0 \neq a_1 \in R$ and $0 \neq b_1 \in \mathrm{Jac}(R)$ such that $a_1 b_1 = r$.

If a_1 is invertible, $n_1 R$ is a direct summand isomorphic to M as before, and we are done again. If not, $a_1 \in \mathrm{Jac}(R)$ as well.

Since $pp_N(n)$ is generated by $a \mid x \wedge xb = 0$, it follows that $a \mid x \wedge xb = 0 \rightarrow a_1 a \mid x$. Hence from Fact 5.5 we obtain $a \sim a_1 ab = a_1 r$. Then $r \leq a_1 r \sim a \leq ab = r$, hence $a_1 r \sim r$. By Fact 5.6, this yields units u and v such that $r = ua_1 rv$. As clearly $rv \sim r$, (3) therefore implies $ua_1 \not\sim r$, hence $a_1 \not\sim r$.

Next choose $n_2 \in N$ such that $n_2 a_1 = n_1$ and a formula $a_2 \mid x \wedge xb_2 = 0$ generating its pp-type in N, where $a_2 b_2 = r$. We get the following picture.

$$
\begin{tikzcd}
\circ \arrow[d, "a_2"] \\
n_2 \ \circ \arrow[d, "a_1"] \\
n_1 \ \circ \arrow[d, "a"] \\
n \ \circ \arrow[d, "b"] \\
0 \ \circ
\end{tikzcd}
$$

Again, if a_2 is invertible, we finish as before. We conclude by showing that there is no other possibility.

Indeed, if a_2 were not invertible, it would be in $\mathrm{Jac}(R)$. Since $pp_N(n_1)$ is generated by $a_1 \mid x \wedge xb_1 = 0$, we would therefore obtain, as above, $a_1 \mid x \wedge xb_1 = 0 \rightarrow a_2a_1 \mid x$, hence $a_1 \sim a_2a_1b_1 = a_2r$, and thus $r \leq a_2r \sim a_1 \leq a_1b_1 = r$. But this would imply $a_1 \sim r$, contradicting the above. $\square$

For the convenience of the reader, we conclude with a direct proof of (4) $\Rightarrow$ (1), because it is based on an argument that is very useful in applications of Kaplansky's Theorem. First of all, by the latter, we may assume that the module $N \in \mathrm{Add}(M)$ in question is countably generated, say by $n_1, n_2, \ldots$. Now choose a direct summand $M_1 \cong M$ of N containing n_1, and write $n_2 = m_{21} + n_2' \in N = M_1 \oplus N_1$ accordingly. Then choose a direct summand $M_2 \cong M$ of N_1 containing n_2', and write $n_3 = m_{31} + m_{32} + n_3' \in M_1 \oplus M_2 \oplus N_2$. Continuing like this, we obtain direct summands $M_i \cong M$ and a decomposition $N = M_1 \oplus \ldots \oplus M_k \oplus N_k$ such that $n_k \in M_1 \oplus \ldots \oplus M_k$. Then the submodule $\bigoplus_{i=1}^{\infty} M_i$ of N contains all generators n_i and is therefore all of N, as desired.

6. Noetherian rings

Here we answer our main question in the affirmative for hereditary noetherian rings by showing that every pure-projective module over such rings is a direct sum of finitely generated modules.

The main idea is to use the torsion theory generated by singular modules. Recall that a ring R is *right Goldie* if R has finite right uniform dimension and the a.c.c. on right annihilators.

Let R be a semiprime right Goldie ring. Then, by Goldie's theorem, the set of regular elements of R is a right Ore set, and R has a (right) classical quotient ring $Q = Q(R)$. Further, Q is a semisimple artinian ring and, as a left R-module, it is flat (being a union of left modules $Ra^{-1} \cong {}_RR$, where a is a regular element of R).

We say that an element of a module is *torsion* if it is annihilated by a regular ring element. A module is said to be *torsion* if every element of it is. By [10, Prop. 2.3.5], an R-module M is torsion iff it is *singular*, i. e., iff the annihilator of any $m \in M$ is an essential right ideal of R. The torsion elements of M form the *torsion submodule* $T(M)$ of M. Note that $P = M/T(M)$ is a *torsion-free* module, i. e., $T(P) = 0$.

The first goal is to prove that, in pure-projective modules, the torsion part splits off. This is true even over semihereditary semiprime Goldie rings, as we show in the following two lemmas.

Lemma 6.1. *Let M be a finitely generated module over a semihereditary semiprime Goldie ring. Then $M = T(M) \oplus P$ where P is projective.*

Proof. This can be proved as [10, Lemma 5.7.4]. It is worthwhile, however, to emphasize where the left and the right Goldie conditions come in now, which is why we reproduce the proof.

Since $P = M/T(M)$ is finitely generated torsion-free and ${}_RQ$ is flat by the right Goldie condition, tensoring P with Q embeds P (as a right R-module) in Q^n. As R is also left Goldie, any $q \in Q$ can be written as $c^{-1}d$, where $c \in R$ is regular. Multiplying by the common left denominator on the left, we obtain an embedding of P in R^n.

Since R is (right) semihereditary, P is projective, hence the short exact sequence $0 \to T(M) \to M \to P \to 0$ splits. □

Note that semiheriditarity is symmetric for rings of finite uniform dimension [9, Thm. 7.64].

Remark 6.2. *Every torsion-free module N over a semihereditary semiprime Goldie ring is flat. (Indeed, in case that N is finitely generated, it is even projective by the previous proof; for the general case, write N as a direct limit of its finitely generated (projective) submodules.)*

Lemma 6.3. *Let M be a pure-projective module over a semihereditary semiprime Goldie ring. Then $M = T(M) \oplus P$ where P is projective.*

Proof. As M is pure-projective, M is a direct summand of a direct sum $N = \oplus_{i\in I} M_i$ of finitely presented R-modules M_i. Let $\pi : N \to M$ be the map that splits the inclusion $M \subseteq N$.

By the previous lemma, $M_i = T(M_i) \oplus P_i$, where P_i is projective. Then $T(N) = \oplus_{i\in I} T(N_i)$ and hence $N = T(N) \oplus P$, where $P = \oplus_{i\in I} P_i$ is projective.

Clearly $T(M) \subseteq T(N)$ and $\pi(T(N)) \subseteq T(M)$, hence π also splits the inclusion $T(M) \subseteq T(N)$. Since $T(N)$ is a direct summand of N, so is $T(M)$, whence the latter is a direct summand also of M.

Thus $M = T(M) \oplus P$, where $P = M/T(M)$ is a torsion-free module. By the above remark, P is flat. But every flat pure-projective module is projective. $\square$

In order to decompose a pure-projective module into a direct sum of finitely generated submodules, we need to put some additional restrictions on the ring.

Theorem 6.4. *Let M be a pure-projective module over a hereditary noetherian prime ring R. Then M is a direct sum of finitely generated modules.*

Proof. By Lemma 6.3, $M = T(M) \oplus P$, where $T(M)$ is torsion, and P is projective. Since R is right hereditary, by a result of Albrecht, P is a direct sum of finitely generated right ideals of R, see [2, (comments to) Thm. 0.2.9].

As above, $T(M)$ is a direct summand of a direct sum of finitely generated torsion modules. But by [10, Lemma 5.7.4], every finitely generated torsion module over a hereditary noetherian prime ring has finite length. Thus we may invoke the Krull-Remak-Schmidt-Azumaya theorem, in the form of [1, Thm. 26.5] applied to the case where M is countably generated (which suffices, as pointed out before Fact 3.4). $\square$

Corollary 6.5. *Every pure-projective module over a hereditary noetherian ring is a direct sum of finitely generated modules.*

Proof. By [10, Thm. 5.4.6], any hereditary noetherian ring is a direct sum of hereditary noetherian prime rings and an artinian (hereditary) ring. So the proof splits into two respective parts, the former of which is dealt with in the theorem. In the latter, artinian, case, finitely generated modules have finite length, so the Krull–Remak–Schmidt–Azumaya argument works again. $\square$

It is not clear to us to what other noetherian rings this result might be extended. The following example shows at least a limitation for our proof to work.

Example 6.6. *Let R be a local commutative noetherian domain which is not uniserial (the power series ring $F[[x, y]]$ over a field F will do). Consider a non-unit $0 \neq c \in R$ and $a, b \in R$ such that the right ideals aR and bR are incomparable. Let M be the finitely presented module $\langle x, y \mid xac + ybc = 0\rangle$. It is not hard to check that M is indecomposable, see (the proof of)* [6, Thm. 20.42].

Since $(xa+yb)c=0$ *and* $xa+yb\neq 0$ *in* M*, we conclude that* $T(M)\neq 0$*. Indeed,* $xa+yb=0$ *in* M *means that* $xa+yb=(xac+ybc)d$ *in the free module generated by* x *and* y*. Equating coefficients at* x*, we obtain* $a=acd$*, hence* $cd=1$*, a contradiction.*

By similar calculations, $x\notin T(M)$*. Thus* $T(M)$ *is not a direct summand of* M*, and* $M/T(M)$ *is not flat.*

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