pure-injective modules over right noetherian serial rings

Gennadi Puninski

Abstract. We classify pure-injective right modules over right noetherian serial rings.

1. Introduction

The purpose of this paper is to give a complete description of pure-injective modules over a serial right noetherian ring. More precisely, we prove that every pure-injective indecomposable right module $M$ over a serial right noetherian ring $R$ is either injective as $R/\text{Ann}(M)$-module or it is the pure-injective envelope of a finitely presented indecomposable (hence uniserial) module over $R$. Furthermore, every pure-injective right module over $R$ is the pure-injective envelope of a direct sum of indecomposable modules.

The classification of pure-injective (=algebraically compact) modules over a ring $R$ is a classical problem which can be divided in two parts: 1) description of indecomposable pure-injective right modules over $R$; and 2) classification of all pure-injective $R$-modules via reduction to the Case 1).

Kaplansky [10] described all indecomposable pure-injective abelian groups and showed that every pure-injective abelian group contains an indecomposable direct summand. Hence there is no pure-injective superdecomposable (without indecomposable direct summands) abelian groups.

Eklof and Fisher [2] extended this classification to commutative Dedekind domains. They also proved that, over an ‘effectively given’ commutative Dedekind domain $R$, the theory of all $R$-modules is decidable.

The paper was written during a visit of the author to the University of Düsseldorf with financial help of the Heinrich Herzt Foundation. He is grateful to both institutions for their kind hospitality. It is the author’s pleasure to thank R. Wisbauer, O. Kerner, I. Herzog and A. Facchini for their helpful comments.
Ziegler [20] created the machinery for the study of pure-injective modules using model theory. In particular, he described the pure-injective indecomposable modules over a commutative valuation domain. In addition, he proposed a general method to solve decidability questions for modules.

Facchini [4] classified indecomposable pure-injective modules over a commutative valuation ring (with zero divisors) using algebraic methods only. Thus the description of pure-injective indecomposable modules over a commutative Prüfer ring follows from Ziegler’s localization theorem [20].

The existence of a commutative valuation domain with a superdecomposable pure-injective module is evident from Fuchs–Salce [5]. Therefore the general classification problem for pure-injective modules over a commutative valuation domain is difficult and open up to now. From [13] it follows that there exists a superdecomposable pure-injective module over a commutative valuation ring \( R \) if and only if \( R \) does not have Krull dimension (in the sense of [8]).

Now let us consider the case of a uniserial and serial ring. In [14] the pure-injective indecomposable modules over a uniserial ring are classified. In particular, a nice description is obtained over any uniserial ring, for which indecomposable finitely presented modules have local endomorphism ring [15]. This classification also improves the corresponding description given in [4] for the commutative case.

Generalov [6], [7] gave a complete description of pure-injective modules over a noetherian serial ring using the structure theory for such rings [11], [17]. In particular, there are no superdecomposable pure-injective modules over such a ring and every pure-injective indecomposable module is either injective or a pure-injective envelope (=completion in the radical power topology) of an indecomposable finitely presented module.

Eklof and Herzog [3] achieved great progress in describing indecomposable pure-injective modules over arbitrary serial rings. They use effective techniques (similar to [14]) working with endodistributive modules [16]. In particular, in [3] some of the above cited results for commutative and uniserial rings are refined.

In this article the description in [3] of indecomposable pure-injective modules over serial rings is improved by some results which were only known for uniserial rings [14]. We apply this general theory to the case of a serial onesided noetherian ring generalizing results in [6], [7].
2. Definitions

Throughout the paper all rings have an identity and all modules are unitary right modules. A module $M$ over a ring $R$ is uniserial if all its submodules are linearly ordered with respect to inclusion. A serial module is a direct sum of uniserial modules. A ring $R$ is called (uni)serial if both modules $R$ and $R_R$ are (uni)serial. Thus uniserial = local serial for rings. The ring $R$ is Krull-Schmidt [9] if every finitely presented right (and left) module over $R$ is a direct sum of modules with local endomorphism rings.

For a module $M$ we denote by $E(M)$ the injective envelope and by $PE(M)$ the pure-injective envelope. The module $M$ is called self-injective (or $M$-injective [18]), if it is injective with respect to submodules, that is, every morphism $N \to M$, where $N$ is a submodule of $M$ can be extended to a morphism from $M$ to $M$.

If $m \in M$, $r \in R$, then we define $pr_m = \{ s \in R : ms = 0 \}$ and $pl_r = \{ m \in M : mr = 0 \}$. For the definition of pp-formulae $\varphi(\bar{x})$ in $n$ free variables $\bar{x} = (x_1, \ldots, x_n)$ see [12] or [20].

If $R$ is a ring, then a (right) pp-formula $\varphi(\bar{x})$ is a condition $A \div \bar{x}B$ ($A$ divides $\bar{x}B$), that is, there exists $\bar{y}$ such that $\bar{y}A = \bar{x}B$, where $A, B$ are rectangular matrices over $R$. Thus a pp-formula states the solvability of a system of linear equations in an $R$-module.

Let $\varphi(\bar{x})$ and $\psi(\bar{x})$ be pp-formulae. We define $\varphi \to \psi$ ($\varphi$ implies $\psi$), if for every module $M$, and every tuple $\bar{m} \in M$, $M \models \varphi(\bar{m})$ implies $M \models \psi(\bar{m})$.

A set of pp-formulae $p(\bar{x})$ is said to be a pp-type if it is closed with respect to finite conjunctions and implications.

Let $\bar{m}$ be a finite tuple of elements from a module $M$. Then we define the pp-type $pp_M(\bar{m}) = \{ \varphi(\bar{x}) \mid \varphi$ is a pp-formula and $M \models \varphi(\bar{m}) \}$. For the pp-type $p$ let $N(p)$ be its pure-injective envelope. This means that the module $N(p)$ is pure-injective, there is a finite tuple $\bar{n} \in N(p)$ (of suitable length) with $N(p) \models p(\bar{n})$, and for every module $M$ and $\bar{m} \in M$ with $M \models p(\bar{m})$ there exists a splitting monomorphism $f : N(p) \to M$ such that $f(\bar{m}) = \bar{n}$. A pp-type $p$ is called indecomposable, if the module $N(p)$ is indecomposable.

3. First classification theorem

We use the notations of [3]. Let $R$ be a serial ring with a primitive idempotent $e \in R$. The pp-type $p(\bar{x})$ in one variable is called an $e$-type, if
Lemma 3.3. \cite{3}. Let $R$ be a serial ring and $e \in R$ a primitive idempotent. Then every complete $e$-type $p(x)$ (over $\emptyset$) in the theory of modules over $R$ is uniquely determined by the set of pp-formulae $sr \mid xr \in p$, where $s \in Re$, $r \in eR$. Besides, $p$ is indecomposable if and only if $sr \mid xr \in p$ implies $s \mid x \in p$ or $xr = 0 \in p$ for all $s \in Re$, $r \in eR$.

Let $M$ be an indecomposable pure-injective module over a serial ring $R$ and $e \in R$ a primitive idempotent such that $Me \neq 0$. Fix a nonzero $m \in Me$ and set $p = pp_M(m)$, so that $p$ is an $e$-type. Define $I = prm \cap eR = \{r \in eR : mr = 0\}$, $I^* = eR \setminus I$; $J = \{s \in Re : m \notin Ms\}$, and $J^* = Re \setminus J$. Then $I \subset eR$ is a right ideal and $J \subset Re$ is a left ideal of $R$. In view of Proposition 3.1 the module $M = N(p)$ is uniquely determined by the $e$-pair $\langle I, J \rangle$ and type $J^*/I = \{s^* \mid xw \cdot xr = 0 : r \in I, s^* \in J^*\} \cup \{\neg(xr^* = 0 + s \mid x) : r^* \in I^*, s \in J\}$ is consistent.

The following lemma gives in addition to \cite{3} the consistency condition for the type $J^*/I$.

Lemma 3.2. Let $R$ be a serial ring, $e \in R$ a primitive idempotent, $I \subset eR$ a right ideal and $J \subset Re$ a left ideal of $R$. Then type $J^*/I$ is consistent if and only if following condition holds

$$s^*r^* \neq sr^* + s^*r \quad (*)$$

for all $r \in I$, $r^* \in I^*$, $s \in J$, $s^* \in J^*$.

Proof. $\Rightarrow$. Let $(M, m)$ be a realization of the type $J^*/I$ and suppose that $s^*r^* = sr^* + s^*r$ for some $r \in I$, $r^* \in I^*$, $s \in J$, $s^* \in J^*$. Since $s^* \mid x$ and $xr = 0$ belongs to $J^*/I$, it follows that $m \in Ms^*$ and $mr = 0$. Then $mr^* = m(r^* - r) \in Ms^*(r^* - r) = Msr^*$, and therefore $(m - ns)r^* = 0$ for some $n \in M$. Hence $M \models sr^* \mid mr^*$, a contradiction, because obviously $sr^* \mid xr^* = s \mid x + xr^* = 0$.

For the proof of the reverse implication the following lemma will be useful.

Lemma 3.3. (common denominator theorem) Let $A \mid \bar{x}B$, $A_1 \mid \bar{x}B_1$, \ldots, $A_n \mid \bar{x}B_n$ be pp-formulae and consider the implication $A_1 \mid \bar{x}B_1 \ldots \wedge A_n \mid \bar{x}B_n \rightarrow A \mid \bar{x}B$. Then this implication can be realized as a sequence of trivial implications: $A_i \mid \bar{x}B_i \rightarrow A_iV_i \mid \bar{x}B_iV_i \rightarrow A \mid \bar{x}B_iV_i$, and $A \mid
\[ \bar{x}B_1V_1 \cdots w A \mid \bar{x}B_nV_n \to A \mid \bar{x}(B_1V_1 + \cdots + B_nV_n) \to A \mid \bar{x}B, \text{ where} \]
\[ A_iV_i = G_iA \text{ for all } i \text{ and } B_1V_1 + \cdots + B_nV_n = B + GA. \]

Proof. Direct matrix calculations using [12, Ch. 8].

Now we prove \( \Leftarrow \) of Lemma 3.2. Since the modules \( Re \) and \( eR \) are uniserial and the theory of all modules over \( R \) is closed with respect to direct product the inconsistency of type \( J^* / I \) yields \( s^* \mid x \wedge xr = 0 \to s \mid x + xr^* = 0 \vdash sr^* \mid xr^* \) for some \( r \in I, r^* \in I^*, s \in J, s^* \in J^* \). By Lemma 3.3, \( s^* \mid x \to s^*a \mid xa \to sr^* \mid xa \), where

\[ s^*a = bsr^*, \]
\[ xr = 0 \to xr^* = 0 \to sr^* \mid xr^* \text{ and } sr^* \mid xa \overset{w}{\to} sr^* \mid x(a + rb) \to sr^* \mid xrb, \]
\[ a + rb = csr^* + r^*. \]

Now 1 and 2 yield \( bsr^* = s^*a = s^*(csr^* + r^* - rb) = s^*csr^* + s^*r^* - s^*rb \), i.e. \( s^*r^* = (b - s^*c)sr^* + s^*rb \). But \( (b - s^*c)s \in J, rb \in I \), which is impossible.

So, the Lemma 3.2 improves [3] to:

**Theorem 3.4.** (see [3]) Let \( R \) be a serial ring. Then there exists a 1-1 correspondence between the set of isomorphism types for nonzero indecomposable pure-injective right modules over \( R \) and the factor set of set of \( e \)-pairs \( (e \in R \text{ is a primitive idempotent}) \) with condition \((*)\) modulo the following equivalence relation: \( \langle I, J \rangle \sim \langle K, L \rangle \) for \( e \)-pair \( \langle I, J \rangle \) and \( f \)-pair \( \langle K, L \rangle \) if and only if there exist \( 0 \neq u \in fRe, 0 \neq v \in eRf \) such that either a) \( uI = K, Lu = J \); or b) \( vK = I, Jv = L \).

Substituting \( s = r = 0 \) in \((*)\) we obtain

\[ s^*r^* \neq 0 \text{ for all } r^* \in I^*, s^* \in J^*. \] (**)

In [15] it is shown that for a uniserial ring \( R \) the conditions \((*)\) and \((**)\) are equivalent if and only if \( R \) is a Krull-Schmidt ring (=the endomorphism rings of all modules \( R/rR, r \in R \) are local). For example, this equivalence holds for a right (hence left) noetherian uniserial ring.

**Proposition 3.5.** To describe serial rings for which the conditions \((*)\) and \((**)\) (for consistency of \( e \)-pairs) are equivalent.

The answer is unknown even for onesided noetherian rings. We can only prove the following
Lemma 3.6. If the assertions (\ast) and (\ast\ast) are equivalent for all \(e\)-pairs over a serial ring \(R\), then \(R\) is a Krull-Schmidt ring.

Proof. For any ring \(R\) the property of being Krull-Schmidt is left-right symmetric. Thus in view of the Drozd–Warfield theorem [1], [17] it suffices to prove that \(\text{End}_R(eR/rR)\) is local for all \(r \in eR\) and a primitive idempotent \(e \in R\). Since the module \(M = eR/rR\) is generated by \(e\), according to [9], it suffices to prove the indecomposability of \(e\)-type \(p = pp_M(e)\).

Assuming the contrary, we obtain \(sr^* | xr^* \in p\), \(xr^* = 0 \notin p\) and \(s \mid x \notin p\) for some \(r^* \in eR\), \(s \in Re\). Then for \(e\)-pair \(I = rR\), \(J = \text{rad}(Re)\) the condition (\ast\ast) is obviously satisfied.

But from \(sr^* | er^* \in M\) we obtain \(usr^* = r^* + rt\), for some \(u, t \in R\) and \(us \in J\), \(rt \in I\). Then (\ast) is not true for the \(e\)-pair \(\langle I, J \rangle\) and \(s^* = e\). \(\square\)

Lemma 3.7. Let \(R\) be a serial two-sided noetherian ring. Then (\ast) and (\ast\ast) are equivalent.

Proof. Let \(\text{Jac}(R)\) be the Jacobson radical of \(R\). By [17], the intersection of powers \(\text{Jac}(R)^n\) is zero. Let \(s^*r^* = sr^* + s^*r\) and \(s^*r^* \neq 0\). Then \(s^*r^* \in \text{Jac}(R)^n \setminus \text{Jac}(R)^{n+1}\) for some \(n\). Since \(s \notin Rs^*\), it follows that \(s \in \text{Jac}(Rs^*) = \text{Jac}(Rs^*)\) (the last equality holds for any ring \(R\) such that \(R/\text{Jac}(R)\) is \(V\)-ring, see [18]). Therefore \(sr^* \in \text{Jac}^{n+1}\) and analogously \(s^*r \in \text{Jac}^{n+1}\) – a contradiction. \(\square\)

4. Example

Let \(R = \begin{pmatrix} Z(p) & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}\) be a serial right but not left noetherian ring, with \(Z(p)\) the localization of integers \(Z\) at the maximal ideal \(pZ\), and \(\mathbb{Q}\) the rational numbers.

Some description of pure-injective indecomposable modules over \(R\) will follow from the general results of section 5. But it is possible to obtain such a classification by a direct calculation.

Let \(e = e_{11}, f = e_{22}\) be orthogonal idempotents in \(R\) such that \(1 = e + f\).

Case 1. \(e\)-pairs. We have the following possibilities for the right ideal \(I \subset eR\):

\[
I_n = \begin{pmatrix} p^nZ(p) & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}, \ n \geq 1; \ I_q = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix} \text{ and } I_0 = \{0\}.
\]

6
The only left ideals $J \subset Re$ are of the form:

$$J_m = \begin{pmatrix} p^m \mathbb{Z}_p & 0 \\ 0 & 0 \end{pmatrix}, \quad m \geq 1 \text{ or } J_0 = \{0\}.$$  

It is not difficult to check that all possible combination of $e$-pairs are consistent. The identification of $e$-pairs is given by the diagonal action (Theorem 3.4) of the element}

$$0 \neq u \in eRe = \begin{pmatrix} \mathbb{Z}_p & 0 \\ 0 & 0 \end{pmatrix}.$$  

So we obtain the following $e$-pairs with the nonisomorphic pure-injective envelopes:

$$\langle I_0, J_0 \rangle; \quad \langle I_0, J_1 \rangle; \quad \langle I_1, J_0 \rangle; \quad \langle I_1, J_1 \rangle; \quad \langle I_q, J_0 \rangle \text{ and } \langle I_q, J_1 \rangle.$$  

Choose the representations for each such pair. Since $eR$ is a serial module, the module $E(eR)$ is indecomposable. Obviously, it corresponds to the $e$-pair $\langle I_0, J_0 \rangle$.

Since $\text{End}(eR)$ is local, the module $\text{PE}(eR)$ is indecomposable by [9]. It corresponds to the $e$-pairs $\langle I_0, J_n \rangle$, $n \geq 1$ and $\text{PE}(eR) \neq eR$.

Similarly we obtain the correspondence:

$$\langle I_q, J_0 \rangle \rightarrow E(eR/I_q),$$

$$\langle I_q, J_m \rangle, \ m \geq 1 \rightarrow \text{PE}(eR/I_q) \neq eR/I_q,$$

$$\langle I_n, J_0 \rangle, n \geq 1 \rightarrow E(eR/I_n) \text{ and }$$

$$\langle I_n, J_1 \rangle; n \geq 1 \rightarrow \text{PE}(eR/I_n) = eR/I_n,$$

and the modules in the last row are nonisomorphic for different $n \geq 1$.

Case 2. $f$-pairs. Here the right ideal $fR$ is a simple module. Hence for every $f$-pair $\langle K, L \rangle$ we obtain $K = K_0 = \{0\}$. In addition, we have the following possibilities for $L \subset Rf$:

$$L_0 = \{0\}; \quad L_q = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix} \text{ and } L^z = \begin{pmatrix} 0 & p^z \mathbb{Z}_p \\ 0 & 0 \end{pmatrix}, \quad z \in \mathbb{Z}.$$  

The consistency for all possible combinations of $f$-pairs is clear. The identification of $f$-pairs is given by the diagonal action of the element

$$0 \neq v \in fRf = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Q} \end{pmatrix}.$$
Thus we have three \( f \)-pairs with nonisomorphic pure-injective envelopes:

\[ \langle K_0, L_0 \rangle; \quad \langle K_0, L_q \rangle; \quad \langle K_0, L^0 \rangle , \]

where \( L^0 = \begin{pmatrix} 0 & \mathbb{Z}_p \\ 0 & 0 \end{pmatrix} \neq L_0 = \{0\} \).

Consider now the identification of \( e \)- and \( f \)-pairs. Since \( fRe = 0 \), it is enough to consider \( 0 \neq v \in eRf = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix} \). Hence \( I = vK = vK_0 = \{0\} = I_0 \) and \( L = Jv \).

So the pairs \( \langle I_0, J_0 \rangle \) and \( \langle K_0, L_0 \rangle \); \( \langle I_0, J_m \rangle \), \( m \geq 1 \) and \( \langle K_0, L^z \rangle \), \( z \in \mathbb{Z} \) are identical. For example \( E(eR) \cong \mathbb{E}(fR) \).

Finally, the \( f \)-pair \( \langle K_0, L_q \rangle \) is represented by the module \( PE(fR) = fR \), since \( fR \) is a one-dimensional vector space over the ring \( \text{End}(fR) \).

So we have the complete description of pure-injective right indecomposable modules over \( R \). Every such module is either injective or the pure-injective envelope of finitely presented indecomposable module over \( R \). Besides, the condition \((*)\) and \((**)\) from section 2 are equivalent.

5. The second classification theorem

The following assertion gives a concrete representation for a pure-injective module over a serial ring.

**Theorem 5.1.** Let \( M \) be a nonzero faithful indecomposable pure-injective module over a serial ring \( R \). If \( M \) is not injective, then it is direct summand for a pure-injective envelope \( PE(T) \) of a (uniserial faithful) ideal \( T \subseteq fR \), where \( f \in R \) is a primitive idempotent.

**Proof.** Choose a primitive idempotent \( e \in R \) such that \( Me \neq 0 \). Let \( 0 \neq m \in Me \) and \( p = pp_M(m) \) be its indecomposable \( e \)-type. Recall that \( I = \{r \in eR : mr = 0\} \), \( J = \{s \in Re : m \notin Ms\} \) and \( m \) realizes \( J^*/I^* \).

Suppose that \( mr \in E(M)sr \) implies \( msr \in Msr \) for all \( r \in eR \), \( s \in Re \). Then in view of Proposition 3.1 the module \( M = N(p) \) is a direct summand of \( E(M) \), and therefore \( M \) is injective.

Hence \( mr^* \in E(M)sr^* \) and \( mr^* \notin Msr^* \) for some \( r^* \in eR \), \( s \in Re \). In particular \( s \in J \), \( r^* \in I^* \) and \( s \neq 0 \). We shall show that \( M \) is a direct summand of the pure-injective envelope of a right ideal \( T \subseteq fR \). Because the type \( p = pp_M(m) \) is indecomposable, it is sufficient to realize \( p \) in \( T \).

**Step 1.** We may assume that \( s \in fRe \) for some primitive idempotent \( f \in R \).
Let $1 = e_1 + \cdots + e_n$, with $e_i$ orthogonal primitive idempotents in $R$. Then $s = e_1 s + \cdots + e_n s$. As the module $Re$ is uniserial, $Rs = Re_i s$ for some $i \leq n$. Set $f = e_i$. Obviously, the formulae $\bar{s} \mid x$ and $f \bar{s} \mid x$ are equivalent.

Besides, $pr \bar{s} \cap eR = pr (f \bar{s}) \cap eR = I$.

**Step 2.** $m \in E(M)\bar{s}$.

We have to check only, that $\bar{s} r = 0$ implies $mr = 0$ for every $r \in eR$. The module $eR$ is uniserial, hence either $r = r^* u$ or $r^* = ru$ for some $u, v \in R$.

If $r = r^* u$, then $\bar{s} r = 0$ implies $mr = mr^* u \in E(M)\bar{s} r^* u = E(M)\bar{s} r = 0$.

If $r^* = ru$ we obtain $\bar{s} r^* = \bar{s} ru = 0$. Then $\bar{s} r^* \mid mr^*$ in $E(M)$ implies $mr^* = 0$ and $r^* \in I$, a contradiction.

**Step 3.** $pr \bar{s} \cap eR = I$.

Let $r \in pr \bar{s} \cap eR$. From $\bar{s} \mid m \in E(M)$ follows $n \bar{s} = m$ for some $n \in E(M)$. Hence $mr = n \bar{s} r = 0$ in $E(M)$, therefore in $M$. So $pr \bar{s} \cap eR \subseteq I$.

For the converse inclusion let $r \in I$. From $\bar{s} \mid x \notin p$ follows that $m \in (pr r \cap Me) \setminus M\bar{s}$. By [3], $Me$ is a uniserial $\text{End}(Me)$-module. Hence $M\bar{s} \subseteq pr r \cap Me$, i.e. $M\bar{s} r = 0$. Since $M$ is faithful, $\bar{s} r = 0$.

**Step 4.** There exists a primitive idempotent $g \in R$ and a subset $S^* \subseteq J^* \cap gR$ such that: for every $t^* \in J^*$, $s^* \in Rt^*$ holds for some $s^* \in S^*$ (i.e. $s^* \mid x \rightarrow t^* \mid x$).

Let $1 = e_1 + \cdots + e_n$, where $e_i$ are orthogonal primitive idempotents.

As above for every element $s^* \in J^*$ we choose an idempotent $a(s^*) \in \{e_1, \ldots, e_n\}$ such that $Ra(s^*) s^* = Rs^*$. The set of left principal ideals $S = \{Rs^* : s^* \in J^*\}$ is linearly ordered by inclusion. Set $S_i = \{Rs^* \in S : a(s^*) = e_i, i = 1, \ldots, n\}$. Then $S = S_1 \cup \ldots \cup S_n$, therefore some $S_i$ are cofinal in $S$. Then $g = e_i$ and $S^* = S_i$ are desired.

Since the module $Re$ is uniserial, for every $s^* \in S^*$ there exists $g(s^*) \in fRg$ with $g(s^*) s^* = \bar{s}$.

**Step 5.** $ns^* = m$ in $M$ for some $n \in Mg$ implies $pr n \cap gR = pr g(s^*) \cap gR$.

We must show that if $r = gr$ is an element of $gR$, then $nr = 0$ if and only if $g(s^*) r = 0$. Since $gR$ is uniserial we have either $s^* = ru$, or $r = s^* v$ for some $u, v \in R$.

Suppose $s^* = ru$. Then $nr = 0$ implies $m = ns^* = nru = 0$ — a contradiction — and $g(s^*) r = 0$ implies $\bar{s} = g(s^*) s^* = g(s^*) ru = 0$ — another contradiction.

Thus $r = s^* v$, so that $mev = mv = ns^* v = nr$ and $g(s^*) r = g(s^*) s^* v = \bar{s} v = \bar{s} ev$. Hence $nr = 0$ if and only if $ev \in pr m \cap eR$, that is, if and only if $ev \in pr \bar{s} \cap eR$ (Step 3), i.e. if and only if $g(s^*) r = 0$.  

9
Let $T$ be a right ideal of $R$ generated by the elements $g(s^*), s^* \in S^*$. Obviously, $\bar{s} \in T$, $T \subseteq fR$ and $T = \{g(s^*)u : s^* \in S^*, u \in R\}$. Let $q$ denote the $e$-type $q = pp_T(s)$.

In view of Proposition 1 we have to check only

**Step 6.** $p = q$.

It is enough to check the coincidence of $p$ and $q$ on the pp-formulae $sr \mid xr$, $s \in Re, r \in eR$.

If $sr \mid xr \in p$ then $s \mid x \in p$ or $xr = 0 \in p$ (Proposition 3.1). If $s \mid x \in p$ we have $s \in J^*$, hence we can choose $s^* \in S^*$ such that $s^* \in Rs$. From $g(s^*)s^* = s$ follows $s^* \mid x \in q$, hence $s \mid x \in q$. If $xr = 0 \in p$ then $xr = 0 \in q$ in view of Step 3.

Let $sr \mid xr \in q$, hence $g(s^*)usr = \bar{s}r$ for some $u \in R$. Then $g(s^*)s^* = \bar{s}$ yields $g(s^*)s^*r = \bar{s}r$, hence $s^*r - gusr \in pr g(s^*) \cap gR$. Since $s^* \in S^* \subseteq J^* \cap gR$ we have $s^* \mid m$, hence we can choose $n \in Mg$ such that $ns^* = m$. Then in view of Step 5 we obtain $s^*r - gusr \in pr n \cap gR$. Hence $mr = ns^*r = n[gusr + (s^*r - gusr)] = ngusr \in Msr$ and $sr \mid xr \in p$. □

### 6. Right noetherian serial rings

The following result yields Part 2 of the general classification problem for a serial onesided noetherian ring.

**Proposition 6.1.** Every nonzero pure-injective right module $M$ over a serial right noetherian ring $R$ contains an indecomposable direct summand.

**Proof.** Let $e \in R$ be a primitive idempotent such that $Me \neq 0$. Then the set $S = \{pr n \cap eR : 0 \neq n \in Me\}$ of right ideals $R$ contains a largest element $pr m \cap eR$, where $0 \neq m \in Me$.

It is enough to show, that the $e$-type $p = pp_M(m)$ is indecomposable, i.e. (Proposition 3.1) $sr \mid xr \in p$ implies $s \mid x \in p$ or $xr = 0 \in p$ for all $s \in Re, r \in eR$.

We have $nsr = mr$ for some $n \in M$, i.e. $(ns - m)r = 0$. If $k = ns - m \neq 0$, then $k \in Me$ and $r \in pr k \cap eR$. Thus $r \in pr m$ and $mr = 0$.

Otherwise $ns = m$ and $s \mid x \in p$. □

**Question 6.2.** Is Proposition 6.1 true for a serial ring with right Krull dimension? (By [19] $R$ also have (the same) left Krull dimension). In particular, is Proposition 6.1 true for left modules over right noetherian serial ring?

Now let us formulate the general classification theorem.
Theorem 6.3. Every pure-injective right module $M$ over a serial right noetherian ring $R$ is a pure-injective envelope $PE(\bigoplus_{i \in I} M_i)$, with indecomposable pure-injective $M_i$. Furthermore, every $M_i$ is either injective as $R/\text{Ann}(M_i)$-module (hence self-injective), or the pure-injective envelope of a finitely presented indecomposable (hence uniserial) module over $R$.

Proof. The first fact follows from Proposition 6.1. For the second let $M$ be an indecomposable pure-injective module and $R = R/\text{Ann}(M)$. If $M$ is not an injective $R$-module, then, in view of Theorem 5.1, $M$ is direct summand of $PE(T)$ for some uniserial right ideal $T$ of $R$. Since $R$ is right noetherian, $T$ is finitely presented and noetherian. Because every epimorphism of noetherian module is isomorphism and module $T$ is uniserial it has a local endomorphism ring. Hence in view of [9] $PE(T)$ is an indecomposable $R$-module and $M \cong PE(T)$. \hfill \Box

References


RUSSIAN STATE SOCIAL UNIVERSITY, LOSINOOSTROVSKAYA, 24, 107150, MOSCOW, RUSSIA