The indices of central simple algebras over function fields of projective spaces over $P_{n,r}$-fields [1]

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Abstract. Let K be a field of characteristic zero and Br K its Brauer group. Bounds for the indices of elements of Br K depending on their exponents are obtained for function fields K of projective spaces and projective curves defined over a $P_{n,r}$-field.

Bibliography: 18 titles.

§1. Introduction

Let K be an arbitrary field of characteristic zero, and Br K its Brauer group. Two numerical invariants, the exponent $e(a)$ and the index $i(a)$, are associated with each element $a \in$ Br K. It is well known that $e(a)$ is a divisor of $i(a)$ and a classical result of Brauer states that $i(a)$ and $e(a)$ have the same sets of prime divisors [1], [2]. Studying the dependence of the index on the exponent is an important direction in the theory of Brauer groups. Unfortunately, there are no general relations here, apart from the one mentioned above, since the dependence between the exponent and the index changes capriciously as K changes. In this situation it is natural to give a concrete statement of the above-mentioned problem.

Given a certain class of fields, find the dependence between the exponents and indices of elements of the Braver groups of fields in this class.

Several important results have been obtained in this direction. For example, it is known that $e(a) = i(a)$ in the case of the class of finite fields, or even C_1-fields (these are classical theorems of Wedderburn and Tsen).

The local and global theories of class fields show that the above equality holds also in the cases of the classes of local or global fields. The coincidence of the exponent and the index has recently been established also in the cases of fields of repeated as well as double formal power series with coefficients in an algebraically closed field (see [3], [4]), as a part of the general conjecture on the coincidence of the exponent and index of elements of Brauer groups over C_2-fields. However, already in the first half of the last century it was shown (Brauer [5]) that the exponent does not always coincide with the index and therefore an answer to the question of their dependence cannot be expressed as a coincidence theorem. On the other hand, a consideration of fields of rational functions in infinitely many variables shows that in the class of such fields for a fixed exponent there can exist elements of arbitrarily large index. Finally, if we consider function fields K on algebraic varieties defined over an arbitrary field k, then the same phenomenon of unboundedness of the indices for a fixed exponent can arise due to elements of the group Br k. Thus, if we are interested in the problem of boundedness of the indices for a fixed exponent for the class of function fields on algebraic varieties, which in many respects is

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AMS 2000 Mathematical Subject Classification. Primary 14F22; Secondary 11R58, 14H05 one of the most important classes of fields, then it is natural to consider such varieties defined over a field k for which the corresponding problem of boundedness of the indices has an affirmative solution. A striking example of the fruitfulness of this approach is given by the recent result of Saltman [6], [7], who showed that if K is a function field on a smooth projective curve defined over the p-adic number field, then for any positive integer n coprime to p, if $e(a) = n$ then the index $i(a)$ is a divisor of n^2. Moreover, this bound cannot be strengthened in the sense that there exist algebras of exponent n and index n^2.

Nevertheless, if the field k is sufficiently arbitrary, then it is difficult to expect to obtain bounds of such a precise type. Until recently one of the main methods of obtaining bounds for the indices in the general situation was the method of 'controlled' extension of the scalars 'killing' the ramification followed by an estimation of the indices of the unramified elements. Important for this latter stage is a result of van den Bergh [8] giving a bound for the indices of unramified elements.

In the present paper we consider the class of purely transcendental extensions K of finite dimension over the so-called $P_{n,r}$-fields and show that the indices of elements of the corresponding Brauer groups depend only on the field of definition, the exponent, and the ramification of a given element. More precisely, we show that in fact the index of an element depends not on the ramification at the given point, but on the point itself where this element is ramified. This result allows us to obtain bounds for the indices that are better than the bounds arising from the results of van den Bergh. Among earlier results on the Brauer groups of function fields of projective spaces, from which bounds for the indices of their elements can be easily extracted (however, not very precise ones), we should mention the paper [9] on the Brauer group of a complex projective space. We also mention the papers [10], [11], where a description of unramified elements of exponent 2 of the Brauer groups of function fields of special curves was obtained, and [12], [13] describing the Brauer groups of special surfaces.

In this paper we successively consider the cases of a projective line, a projective space, and, finally, the case of a smooth projective curve.

§2. Preliminary results

We begin with the notation and conventions required in what follows.

Let k be an arbitrary field of characteristic zero containing a primitive nth root of unity, X a smooth projective variety over k, $K = k(X)$ its function field. For a discrete valuation ring R with residue field $k(R)$ such that the field of fractions of R coincides with K there exists the homomorphism of ramification at R

$$\partial_R : \mathrm{Br}\ K \longrightarrow \mathrm{Hom}_{cont}(G_R, \mathbb{Q}/\mathbb{Z}) = H^1(G_R, \mathbb{Q}/\mathbb{Z}),$$

where $G_R = \mathrm{Gal}(\overline{k}/k(R))$ and $\overline{k}$ is the algebraic closure of k (see. [14]). A central simple algebra (c.s.a.) A over K is said to be *ramified* at R, if $\partial_R([A]) \neq 0$; then R is called a *ramification point.* It is known that the number of ramification points of a c.s.a. is finite. The subgroup $\cap_R \ker \partial_R$, where R runs over the set of rings with the aforementioned properties, is called the *unramified Brauer group of the field* K and denoted by $\mathrm{Br}_{nr}K$.

The homomorphism ∂_R has the following structure (see. [14]). Let R' and K' be the completion of the ring R and the field K, respectively. Then R' and K' can be identified with the ring $k(R)[[\pi]]$ and the field $k(R)((\pi))$, respectively. Let K'_{nr} be the maximal unramified extension of the field K'. Then K'_{nr} coincide with the field $\overline{k}((\pi))$, $\mathrm{Gal}(K'_{nr}/K') = G_R$, and $\overline{K'}$ is the union of all fields of the form $\overline{k}((\pi^{1/n}))$. Then the Galois group $\mathrm{Gal}(\overline{K'}/K_{nr})$ is isomorphic to the group $\varprojlim \mathbb{Z}/n$. The cohomological dimension of this group is equal to 1; consequently, $H^2(\mathrm{Gal}(\overline{K'}/K'_{nr}), \overline{K'}^*) = 0$. Since $H^1(\mathrm{Gal}(\overline{K'}/K_{nr}), \overline{K'}^*) = 0$ by Hilbert's Theorem 90, the sequence of restriction and inflation gives the isomorphism

$$H^2(\mathrm{Gal}(K'_{nr}/K'), {K'_{nr}}^*) \cong H^2(\mathrm{Gal}(\overline{K'}/K'), \overline{K'}^*). \tag{1}$$

The valuation $v : K' \longrightarrow \mathbb{Z}$ can be naturally extended to a valuation $v : K'_{nr} \longrightarrow \mathbb{Z}$, which is a homomorphism of $\mathrm{Gal}(K'_{nr}/K')$-modules ($\mathbb{Z}$ is regarded as a discrete module). Then we obtain a homomorphism

$$H^2(\mathrm{Gal}(K'_{nr}/K'), {K'_{nr}}^*) \longrightarrow H^2(\mathrm{Gal}(\overline{K}'_{nr}/K'), \mathbb{Z}) = H^2(G_R, \mathbb{Z}). \tag{2}$$

The exact sequence of G_R-modules $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ gives the isomorphism $H^2(G_R, \mathbb{Z}) \cong H^1(G_R, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{cont}(G_R, \mathbb{Q}/\mathbb{Z})$.

Combining the isomorphism (1) and the homomorphism (2) we obtain a homomorphism

$$\partial_{R'} : \mathrm{Br}\ K' \longrightarrow \mathrm{Hom}_{cont}(G_R, \mathbb{Q}/\mathbb{Z}).$$

Then the homomorphism ∂_R is defined as the composite

$$\mathrm{Br}\ K \xrightarrow{\mathrm{res}} \mathrm{Br}\ K' \xrightarrow{\partial_{R'}} \mathrm{Hom}_{cont}(G_R, \mathbb{Q}/\mathbb{Z}).$$

If K is a field containing a primitive nth root of unity ρ and $a, b \in K^*$, then the *symbol-algebra* $(a, b)_n$ is by definition the cyclic algebra generated over K by elements α, β such that $\alpha^n = a$, $\beta^n = b$, $\alpha\beta = \rho\beta\alpha$. The element of the Brauer group containing the algebra $(a, b)_n$ is denoted by $[(a, b)_n]$.

Using the description of the homomorphism ∂_R one can easily prove the following assertion.

Lemma 1 *Suppose that π, a, and b are, respectively, a prime element and units of the valuation related to the ring R. Then*
(i) $\partial_R([(a, \pi)_n]) = 0$ if and only if $a \in (K^*)^n$;
(ii)$\partial_R([(a, b)_n]) = 0$.

We note that in the case of a projective space $\mathbb{P}^m_k$ there exists a one-to-one correspondence between irreducible unitary (in the sense of the lexicographical ordering) polynomials and local rings R of finite points of codimension 1.

§3. The case of a projective line and $P_{n,r}$-fields

First we consider the case where X is the projective line $\mathbb{P}^1_k$. In this case, $\mathrm{Br}\ k = \mathrm{Br}_{nr}\ k(\mathbb{P}^1_k)$. Let $k(\mathbb{P}^1_k) = k(x)$. If $\mathcal{A}$ is a c.s. $k(\mathbb{P}^1_k)$-algebra and $f(x) \in K[x]$ is an irreducible polynomial defining a valuation of the field $k(\mathbb{P}^1_k)$, then we set $\mathcal{A}_{f(x)} = \mathcal{A} \otimes k(\theta)((f))$, where

θ is a root of the polynomial $f(x)$ and $k(\theta)((f(x)))$ is the field of formal power series (the completion of the field $k(\mathbb{P}^1_k)$ relative to the valuation defined by the polynomial $f(x)$).

In addition, the ramification of algebras satisfies the following reciprocity law of Faddeev.

Proposition 2 *([15], [16, III, Prop. 2.1], [17, §1.2]) There exists the following exact sequence*

$$0 \to \mathrm{Br}\ k \to \mathrm{Br}\ k(\mathbb{P}^1_k) \xrightarrow{\oplus\partial_R} \bigoplus_R H^1(G_R, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{cor}} H^1(\mathrm{Gal}(\overline{k}/k), \mathbb{Q}/\mathbb{Z}) \to 0,$$

where R runs over the set of discrete valuation rings such that their fields of fractions coincide with $k(\mathbb{P}^1_k)$, while cor *sums the local corestriction maps.*

We note that it follows from Faddeev's reciprocity law that if two c.s.a.'s over $k(\mathbb{P}^1_k)$ have identical ramification at the rings corresponding to the finite points of the projective line, then they have identical ramification also at the ring corresponding to the infinite point. In addition, if $\mathcal{A}$ is unramified, then $\mathcal{A}$ is similar to some constant c.s.a., that is, an algebra of the form $\mathcal{B} \otimes_k k(\mathbb{P}^1_k)$, where $\mathcal{B}$ is a c.s.a. over k.

We shall need the following definition

Definition 3 *Two c.s. $k(\mathbb{P}^1_k))$-algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be Faddeev equivalent if there exists a c.s. k-algebra $\mathcal{C}$ such that*

$$\mathcal{A} \sim \mathcal{B} \otimes_{k(\mathbb{P}^1_k)} (\mathcal{C} \otimes_k k(\mathbb{P}^1_k)).$$

Clearly, this is an equivalence relation. Algebras $\mathcal{A}$ and $\mathcal{B}$ are Faddeev equivalent if and only if they have identical ramification.

Our main result in the case of a projective line based on the following construction. As already noted, all the finite ramification points of an algebra $\mathcal{A}$ correspond to irreducible unitary polynomials $f_i(x)$ over k. Let $F(x)$ be the product of all such polynomials $f_i(x)$. Our first result is the following theorem

Theorem 4 *Suppose that k is a field containing a primitive nth root of unity and $\mathcal{A}$ is a c.s.a. over $k(\mathbb{P}^1_k))$ of exponent n. Then $\mathcal{A}$ is Faddeev equivalent to a c.s.a. of index at most $n^{[(\deg\ F(x)+1)/2]}$, where $[(\deg\ F(x)+1)/2]$ is the integer part of the number $(\deg\ F(x)+1)/2$.*

Proof. Let $m = [(\deg\ F(x)+1)/2]$. We prove the theorem by induction on m. If $m = 0$, then $\deg\ F(x) = 0$ and $\mathcal{A}$ is unramified at the finite points. Then by the Faddeed reciprocity law the algebra $\mathcal{A}$ is unramified. Thus, $\mathcal{A}$ Faddeev equivalent to the trivial algebra.

Suppose that the theorem is true for $m < s$ and consider the case $m = s$. We observe that for $\deg\ F(x) = 1$ the algebra $\mathcal{A}$ is Faddeev equivalent to an algebra of the form $(u, x-a)_n$, where $F(x) = x - a$ and $u \in k^*$. Thus, we assume that $\deg\ F(x) > 1$.

First we suppose that $F(x)$ is a product of linear polynomials. Let $(x-a_1)(x-a_2)$ divide $F(x)$. Then

$$\mathcal{A}_{x-a_i} \sim \mathcal{B}_i \otimes (u_i, x-a_i)_n \otimes_{k(\mathbb{P}^1_k)} k((x-a_i)), i = 1, 2,$$

where u_i is a non-zero element of the field k and $\mathcal{B}_i$ is some c.s.a. over k. Then the algebra

$$\mathcal{C} = \mathcal{A} \otimes_{k(\mathbb{P}^1_k)} ((x-a_1)u_2^{-1}/(a_2-a_1), (x-a_2)u_1/(a_1-a_2))_n$$

is unramified at the polynomials $x - a_1$ and $x - a_2$. Indeed,

$$\mathcal{C}_{x-a_1} = \mathcal{B}_1 \otimes (u_1, x-a_1) \otimes ((x-a_1)u_2^{-1}/(a_2-a_1), u_1)_n \sim$$

$$\mathcal{B}_1 \otimes (u_1, x-a_1) \otimes ((x-a_1), u_1)_n \otimes (u_2^{-1}/(a_2-a_1), u_1)_n \sim$$

$$\mathcal{B}_1 \otimes (u_1, x-a_1) \otimes (u_1^{-1}, x-a_1))_n \otimes (u_2^{-1}/(a_2-a_1), u_1)_n \sim$$

$$\mathcal{B}_1 \otimes (u_2^{-1}/(a_2-a_1), u_1)_n.$$

It can be shown in similar fashion that

$$\mathcal{C}_{x-a_2} \sim \mathcal{B}_2 \otimes (u_2^{-1}, u_1/(a_1-a_2))_n.$$

Thus, all the finite ramification points of the algebra $\mathcal{C}$ correspond to irreducible unitary divisors of the polynomial $F(x)/(x-a)(x-b)$. Since

$$[(\deg\ F(x)/(x-a)(x-b)+1)/2] = [(\deg\ F(x)-2+1)/2] = s-1,$$

by the induction hypothesis the algebra $\mathcal{C}$ is Faddeev equivalent to an algebra of index at most n^{s-1}. Then $\mathcal{A}$ is Faddeev equivalent to an algebra of index at most n^s.

We now suppose that $F(x)$ has an irreducible unitary divisor $f(x)$ of degree greater than 1.

We have

$$\mathcal{A}_{f(x)} = \mathcal{A} \otimes_{k(\mathbb{P}^1_k)} k(\mathbb{P}^1_k)_{f(x)} \sim \mathcal{B} \otimes_{k(\mathbb{P}^1_k)_{f(x)}} \otimes (\alpha, f(x))_n,$$

where α is some non-zero element of the field $k(\theta)$, θ is a root of the polynomial $f(x)$, and $\mathcal{B}$ is a c.s.a. over $k(\theta)$. There exists a polynomial $g(x) \in k[x]$ such that $\alpha = g(\theta)$ and $\deg\ g(x) < \deg\ f(x)$.

Then $\mathcal{A} \otimes_{k(\mathbb{P}^1_k)} (g(x)^{-1}, f(x))_n$ is unramified at $f(x)$. If $\deg\ g(x) = 0$, then the finite ramification points of the latter algebra correspond to irreducible unitary divisors of the polynomial $F(x)/f(x)$. Since $[(\deg\ F(x)/f(x)+1)/2] < s$, it follows that the latter algebra is Faddeev equivalent to an algebra of index at most n^{s-1}. Then $\mathcal{A}$ is Faddeev equivalent to an algebra of index at most n^s.

In the case $\deg\ g(x) > 0$ we denote by $g_0(x)$ an irreducible unitary divisor of the polynomial $g(x)$ of positive degree, so that $g(x) = g_0(x)^m g_1(x)$, where $g_0(x)$ and $g_1(x)$ are coprime. If the algebra $\mathcal{A}$ is ramified at the polynomial $g_0(x)$, then the finite ramification points of the algebra $\mathcal{C}$ correspond to some irreducible unitary divisors of the polynomial $F(x)g_1(x)/f(x)$. Since $\deg\ g_1(x) < \deg\ g(x) < \deg\ f(x)$, we have $[(\deg\ F(x)g_1(x)/f(x)+1)/2] < s$. Consequently, $\mathcal{A}$ is Faddeev equivalent to an algebra of index at most n^s.

If, however, the algebra $\mathcal{A}$ is unramified at the polynomial $g_0(x)$, then let $h(x)$ be the remainder when $f(x)$ is divided by $g_0(x)$. Then

$$\mathcal{A} \otimes (g(x)^{-1}, f(x))_n \otimes (g(x)^{-1}, h(x)^{-1})_n \sim \mathcal{A} \otimes_{k(\mathbb{P}^1_k)} (g(x)^{-1}, f(x)h(x)^{-1})_n$$

and the finite ramification points of the latter algebra correspond to some irreducible unitary divisors of the polynomial $F(x)g_1(x)h(x)/f(x)$. Since

$$\deg\ g_1(x) + \deg\ h(x) < \deg\ g_1(x) + \deg\ g_0(x) \leq \deg\ g(x) \leq \deg\ f(x) - 1,$$

we have $[(\deg\ F(x)g_1(x)/f(x) + 1)/2] < s$. Consequently, $\mathcal{A}$ is Faddeev equivalent to an algebra of index at most n^s. The theorem is proved.

Remark. Note that in the case $n = 2$ we have $[(\deg\ F(x) + 1)/2] = g + 1$, where g is the genus of the hyperelliptic curve corresponding to the equation $y^2 = F(x)$.

Thus, in the case $n = 2$ the preceding theorem can be reformulated as follows.

Theorem 5 *Suppose that $\mathcal{A}$ is a c.s.a. over $k(\mathbb{P}^1_k))$ of exponent 2. Then $\mathcal{A}$ is Faddeev equivalent to a c.s.a. of index at most n^{g+1}, where g is the genus of the hyperelliptic curve corresponding to the equation $y^2 = F(x)$.*

Remark. The proof of theorem 4 can be used for obtaining the following result of Bloch [18].

Theorem 6 *Suppose that k is a field containing the group μ_n of nth roots of 1 and $L = k(x_1, ..., x_s)$ is the function field of the space $\mathbb{P}^s_k$. Then the natural homomorphism induced by the norm residue homomorphism*

$$R_{n,k} : K_2(k)/nK_2(k) \longrightarrow {}_n\mathrm{Br}\ k \quad \textit{and} \quad R_{n,L} : K_2(L)/nK_2(L) \longrightarrow {}_n\mathrm{Br}\ L$$

have isomorphic cokernels.

Proof. It is easy to see that it suffices to prove the theorem in the case $L = k(x)$. We construct a homomorphism

$$\phi : {}_n\mathrm{Br}\ k/\mathrm{im}(R_{n,k}) \longrightarrow {}_n\mathrm{Br}\ k(x)/\mathrm{im}(R_{n,k(x)}),$$

by setting $\phi(a + \mathrm{im}(R_{n,k})) = \mathrm{res}_{k(x)/k}(a) + \mathrm{im}(R_{n,k(x)})$. We now check the correctness of this definition. Let $a \in \mathrm{im}(R_{n,k})$; then a can be represented as a product of symbol-algebras. Consequently, $\mathrm{res}_{k(x)/k}(a)$ can be represented as a product of symbol-algebras and $\mathrm{res}_{F/k}(a) \in \mathrm{im}(R_{n,F})$. The correctness is proved.

We claim that the homomorphism ϕ is injective. Let $\mathrm{res}_{k(x)/k}(a) \in \mathrm{im}(R_{n,k(x)})$. Then $\mathrm{res}_{k(x)/k}(a)$ can be represented as a product of symbol-algebras. Consequently, $\mathrm{res}_{k(x)_\infty/k}(a)$ can be represented as a product of symbol-algebras, where $k(x)_\infty$ is the completion of the field $k(x)$ relative to the valuation at the infinite point. Hence, $a \in \mathrm{im}(R_{n,k})$. The injectivity is thus proved. We claim that ϕ is surjective. Any element b of ${}_n\mathrm{Br}\ F$ can be represented as an algebra of the form $(\mathcal{A} \otimes_k k(x)) \otimes (f_i, g_i)$, where $\mathcal{A}$ is some algebra over k and $f_i, g_i \in k(x)$. Then $b + \mathrm{im}(R_{n,k(x)})$ is the image of $\mathcal{A} + \mathrm{im}(R_{n,k})$. Consequently, the cokernels are isomorphic. The theorem is proved.

As a simple corollary of the preceding theorem we obtain that a s.c.a. of exponent n over $k(x_1, ..., x_s)$ is similar to a tensor product of cyclic algebras if and only if this is true for any c.s. k-algebra of exponent dividing n. In the case of special fields k this gives an elementary proof of Merkur'ev-Suslin theorem for fields $k(x_1, ..., x_s)$.

Theorem 4 gives upper bounds for the indices of c.s. algebras of exponent n over $k(\mathbb{P}^1_k)$ modulo indices of s.s. k-algebras of exponent dividing n. In the case of special classes of fields of constants k Theorem 4 gives bounds for the indices of c.s. algebras of exponent n over $k(\mathbb{P}^1_k)$. The class of $P_{n,r}$-fields is important in this respect.

Definition 7 *Let n, r be non-negative integers, $n > 0$. A field k is called $P_{n,r}$-field if for any k-algebra $\mathcal{A}$ of exponent dividing n the index $i(\mathcal{A})$ divides n^r.*

Note that if k is a $P_{n,r}$-field, then k is a $P_{n,s}$-field for any $s > r$.

We now give several elementary properties of $P_{n.r}$-fields for convenience of further exposition.

Lemma 8 *Let $n = p_1^{\alpha_1}, \ldots, p_s^{\alpha_s}$, where the p_i are distinct primes and $\alpha_i \geq 1$. If k is a $P_{n,r}$-field, then k is a $P_{p_i^{\alpha_i},r}$-field.*

Proof. Let $\mathcal{A}$ be any algebra of exponent dividing $p_i^{\alpha_i}$. Then $i(\mathcal{A})$ divides n^r. Consequently, $i(\mathcal{A})$ divides $p_i^{\alpha_i r}$.

Lemma 9 *Let $n = p_1^{\alpha_1}, \ldots, p_l^{\alpha_l}$, where the p_i are distinct primes and $\alpha_i \geq 1$. If k is a $P_{p_i^{\alpha_i},s_i}$-field, then k is a $P_{n,r}$-field, where $r = LCM(s_1, \ldots, s_l)$.*

Proof. Suppose that $e(\mathcal{A})$ divides n and $\mathcal{A} = \mathcal{A}_1 \otimes, \ldots, \otimes \mathcal{A}_s$ is the decomposition of the algebra $\mathcal{A}$ into a tensor product of algebras of exponents dividing $p_i^{\alpha_i}$. Then $i(\mathcal{A}) = \prod_i i(\mathcal{A}_i)$ and $i(\mathcal{A}_i)$ divides $p_i^{\alpha_i s_i}$. Consequently, $i(\mathcal{A})$ divides $\prod_i p_i^{\alpha_i s_i}$. Then $i(\mathcal{A})$ divides $\prod_i p_i^{\alpha_i r}$.

Then preceding lemmas allow us to restrict our attention to primary n in the study of the property $P_{n,r}$. In what follows, $n = p^m$, where p is a prime.

Lemma 10 *Suppose k/F is a finite field extension and k is a $P_{n,r}$-field. Then F is a $P_{n,s}$-field, where $s = [log_n[k : F]_p] + 1 + r$ and $[k : F]_p$ is the p-part of the integer $[k : F]$.*

Proof. Suppose that $e(\mathcal{A})$ divides n. Then $e(\mathcal{A} \otimes_F k)$ divides n which in turn means that $i(\mathcal{A} \otimes_F k)$ divides n^r and $i(\mathcal{A})$ divides $[k : F]_p n^r$. Consequently, F is a $P_{n,s}$-field, where $s = [log_n[k : F]_p] + 1 + r$.

Remark. The latter lemma gives a useful method for constructing new $P_{n,r}$-fields from given ones.

In conclusion we give a list (which is far from complete) of some fields of this type.

(i) Algebraically closed fields and finite fields are $P_{n,0}$-fields.

(ii) Real closed fields are $P_{2,1}$-fields and $P_{n,0}$-fields if n is odd.

(iii) The function field of a curve over an algebraically closed field is a $P_{n,0}$-field.

(iv) The function field of a curve over a real closed field is a $P_{2,1}$-field and $P_{n,0}$-field if n is odd.

(v) The function field of a local curve is a $P_{n,2}$-field in the case where n is coprime to the characteristic of the residue field (see [6], [7]).

(vi) Let R be a Henselian discrete valuation field with field of fractions F and residue field k. Suppose that n is coprime to the characteristic of the field k and the Brauer group of any projective curve over every finite extension of the field k is trivial. Then for any curve C over K the field $K(C)$ is a $P_{n,2}$-field (see [6], [7]).

(vii) Suppose that k is a field with Henselian valuation of rank m and residue field of characteristic zero which is a $P_{n,r}$-field. Then k is a $P_{n,m+r}$-field.

(viii) Suppose that R is an excellent Henselian local ring of dimension two, K is its field of fractions, and k is the residue field. Suppose that k is separably closed and n is coprime to the characteristic of the field k. Then K is a $P_{n,1}$-field (see [4]).

(ix) A C_2-field is a $P_{n,1}$-field for $p \in \{2,3\}$.

(x) Suppose that k contains a primitive nth root of 1, where n is coprime to the characteristic of k, and the group $k^*/(k^*)^n$ is finite. Then k is a $P_{n,r}$-field for some r.

For $P_{n,r}$-fields Theorem 4 can be restated as follows.

Theorem 11 *Suppose that k is a $P_{n,r}$-field containing a primitive nth root of 1, $\mathcal{A}$ is a c.s.a. over $k(\mathbb{P}^1_k)$ of exponent n and $F(x)$ is the same as above. Then $i(\mathcal{A}) \leq n^{r+[(deg\ F+1)/2]}$.*

Generally speaking, the preceding bound is not optimal, which follows, for example, from a result of Saltman ([6], [7]), where k is a finite extension of the field $\mathbb{Q}_p$.

However, the situation is not that simple if the whole class of $P_{n,r}$-fields is considered. Moreover, the following question seems to us to be meaningful.

Question. *Let n, r be positive integers. Does there exist a $P_{n,r}$-field, containing primitive nth root of 1 such that the bound obtained in Theorem 11 is attained?*

§4. The case of a projective space

Theorem 4 can be generalized to the case of a projective space.

Theorem 12 *Suppose that k is a $P_{n,r}$-field containing a primitive nth root of 1, $L = k(\mathbb{P}^s_k) = k(x_1, ..., x_s)$, and $\mathcal{A}$ is a c.s.a. over L of exponent n ramified at polynomials $F_1(x_1, ..., x_s)$,..., $F_t(x_1, ..., x_s)$. Let*

$$G(x_1, ..., x_s) = F_1(x_1, ..., x_s)...F_t(x_1, ..., x_s)$$

and let d_j be the degree of the polynomial $G(x_1, ..., x_s)$ in the variable x_j. Then

$$i(\mathcal{A}) \leq n^{r+M_1+...+M_s+min_i\{\sum_{1\leq j\leq s, j\neq i} M_j\}},$$

where $M_i = [(d_i + 1)/2]$.

We preface the proof of the theorem with two lemmas.

Lemma 13 *Suppose that $\mathcal{A}$ is a c.s.a. over $L = k(\mathbb{P}^s_k) = k(x_1, ..., x_s)$ with finite ramification at irreducible polynomials $F_1(x_1, ..., x_s)$,..., $F_t(x_1, ..., x_s)$. Then the finite ramification of $\mathcal{A}$ regarded as an algebra over*

$$k(x_1, ..., x_{s-1})(\mathbb{P}^1_k) = k(x_1, ..., x_{s-1})(x_s)$$

(that is, over the field of rational functions with field of constants $k(x_1, ..., x_{s-1})$) occurs only at some of the unitary polynomials that are divisors of the polynomial

$$F_1(x_1, ..., x_s) \dots F_t(x_1, ..., x_s)$$

regarded as a polynomial in $k(x_1, ..., x_{s-1})[x_s]$.

Proof. We consider $\mathcal{A}$ as an algebra over $k(x_1, ..., x_{s-1})(x_s)$. let

$$H(x_s) \in k(x_1, ..., x_{s-1})[x_s]$$

be a unitary irreducible polynomial at which $\mathcal{A}$ is ramified. Then

$$H(x_s) = x_s^n + (u_{n-1}/v_{n-1})x_s^{n-1} + ... + u_0/v_0,$$

where $u_i, v_i \in k[x_1, ..., x_{s-1}]$ and u_i/v_j are irreducible fractions.

The algebra $\mathcal{A}$ is similar to a product of some cyclic algebras, that is,

$$\mathcal{A} \sim (H(x_s), G)_l \otimes \mathcal{B},$$

where G is some element of the ring $k(x_1, ..., x_{s-1})[x_s]$ and $\mathcal{B}$ is a product of symbol-algebras not containing the element $H(x_s)$. The algebra $\mathcal{A}$ is ramified at $H(x_s)$ if and only if

$$G(\theta) \notin (k(x_1, ..., x_{s-1})(\theta)^*)^l, \tag{3}$$

where θ is a root of the polynomial $H(x_s)$ in an algebraic extension of the field $k(x_1, ..., x_{s-1})$ (see Lemma 1). We claim that

1) the polynomial $T = H(x_s)\ \mathrm{LCM}(v_0, ..., v_{n-1})$ is irreducible over k;

2) the polynomial T contains the ramification of the algebra $\mathcal{A}$ as an algebra over $k(x_1, ..., x_s)$ with field of constants k.

1) Suppose that T is irreducible; then $T = T_1T_2$ where $T_i \in k[x_1, ..., x_s]$. If T_1 and T_2 contain the variable x_s, then the polynomial $H(x_s)$ is reducible as a polynomial in x_s. Thus, we may assume that T_1 does not contain x_s. But then the set of coefficients of the polynomial $T \in k[x_1, ..., x_{s-1}][x_s]$ has an irreducible common divisor in $k[x_1, ..., x_{s-1}]$. We observe that the coefficient of the polynomial T at x_s^n is $\mathrm{LCM}(v_0, ..., v_{n-1})$ and the coefficient at x_s^i is equal to

$$\mathrm{LCM}(v_0, ..., v_{n-1})u_i/v_i,\ i = 0, ..., n-1,$$

Let $S \in k[x_1, ..., x_{s-1}]$ be an irreducible divisor of all coefficients of the polynomial T. Then S is a divisor of $\mathrm{LCM}(v_0, ..., v_{n-1})$. Consequently, S divides at least one of the v_i. We choose a polynomial v_j that is divisible by S to the greatest possible power. Then S divides $\mathrm{LCM}(v_0, ..., v_{n-1})u_j/v_j$. But S does divide $\mathrm{LCM}(v_0, ..., v_{n-1})/v_j$ by the choice of v_j. Then S is a divisor of u_j and therefore u_j and v_j have a common divisor. This is a contradiction with the irreducibility of u_j/v_j. Therefore, T is irreducible.

2) We claim that $\mathcal{A}$ is unramified at T. We have $\mathcal{A} \sim (H(x_s), G)_N \otimes \mathcal{B}$. Then

$$\mathcal{A} \sim (H(x_s)\ \mathrm{LCM}(v_0, ..., v_{n-1})\ \mathrm{LCM}(v_0, ..., v_{n-1})^{l-1}, G)_l \otimes \mathcal{B} =$$

$$(\ \mathrm{LCM}(v_0, ..., v_{n-1})^{l-1}T, G)_l \otimes \mathcal{B}.$$

Since the algebra $\mathcal{B}$ is a product of symbol-algebras not containing $H(x_s)$, we conclude that $\mathcal{B}$ is unramified at T. In addition, $(\text{ LCM}(v_0, ..., v_{n-1}))^{l-1}$ is not divisible by T, since it does not contain the variable x_s. Thus, the ramification at T is determined by the element G. Moreover, the algebra $\mathcal{A}$ is ramified at T if and only if $G \notin (k(C_T)^*)^l$, where C_T is the hypersurface defined by the equation $T = 0$. Suppose $G \in (k(C_T)^*)^l$. Then there exist polynomials $G_1, G_2 \in k[x_1, ..., x_s]$ such that $G = G_1^l + G_2 T$. Substituting in the latter equality the element θ instead of x_s we obtain

$$G(\theta) = G_1(\theta)^l + G_2 T(\theta) = G_1(\theta)^l.$$

But this contradicts (3). Thus, the algebra $\mathcal{A}$ is ramified at T. The lemma proved.

Lemma 14 *Suppose that k is a field of characteristic zero, $H_1, \dots, H_s$ are irreducible pairwise coprime polynomials in $k[x_1, \dots, x_s]$. Then there exist infinitely many elements a of the field k such that the polynomials $\overline{H}_i = H_i(x_1, \dots, x_{s-1}, a) \in k[x_1, \dots, x_{s-1}]$ are pairwise coprime.*

Proof. Let $Y_{ij} \subset \mathrm{A}_k^m$ be the sets defined by the systems of equations $H_i = 0$ and $H_j = 0$, $i \neq j$. Then dim $Y_{ij} \leq s - 2$ and dim $\cup_{i,j} Y_{ij} \leq s - 2$. We consider $\cup_{i,j} Y_{ij} \cap X_a$, where X_a is the hyperplane defined by the equation $x_s = a$. Suppose that there exist polynomials H_i and H_j such that $\overline{H}_i$ and $\overline{H}_j$ have a common divisor $h \in k[x_1, \dots, x_{s-1}]$. Let $Z \subset \mathrm{A}_k^s$ be the set defined by the equation $h = 0$ where h is regarded as a polynomial in $k[x_1, \dots, x_s]$. Then

$$Z \cap X_a \subset Y_{ij} \cap X_a \subset \cup_{i,j} Y_{ij} \cap X_a.$$

Indeed, if $\overline{H}_i = h_i h$ and $\overline{H}_j = h_j h$, then $Y_{ij} \cap X_a$ is defined by the equations

$$x_s = a, h_i h = 0, h_j h = 0.$$

Since $Z \cap X_a$ is defined by the equations $x_s = a$ and $h = 0$, we have $Z \cap X_a \subset Y_{ij} \cap X_a$. Then

$$s - 2 \leq \text{dim } Z \cap X_a = \text{dim } Y_{ij} \cap X_a = \text{dim } \cup_{i,j} Y_{ij} \cap X_a,$$

since dim $Y_{ij} \cap X_a \leq s - 2$. Consequently, dim $\cup_{i,j} Y_{ij} \cap X_a = $ dim $\cup_{i,j} Y_{ij} = s - 2$.

Since dim $\cup_{i,j} Y_{ij} \cap X_a = s-2$, we conclude that X_a contains an irreducible component of the variety $\cup_{i,j} Y_{ij}$ of dimension $s - 2$. Note that the number of such components is finite.

Suppose that for an infinite set of elements $a \in k$ some of the polynomials $\overline{H}_i$ and $\overline{H}_j$ have a common divisor in $k[x_1, \dots, x_{s-1}]$. Then for such an element a the hyperplane X_a contains an irreducible component of the variety $\cup_{i,j} Y_{ij}$ of dimension $s - 2$. Since the number of components is finite, there exist distinct a_1 and a_2 in k such that X_{a_1} and X_{a_2} contain the same component of dimension $s - 2$. But $X_{a_1} \cap X_{a_2} = \emptyset$. The contradiction thus obtained proves the lemma.

Proof of Theorem 12. We apply induction on s. In the case $s = 1$ the assertion follows from Theorem 11. Suppose that the assertion is true for $s \leq m - 1$ and consider the case $s = m$.

The algebra $\mathcal{A}$ is an algebra over the function field of the projective line $\mathbb{P}^1_{k(x_1,\dots,x_{m-1})}$. By Lemma 13 all finite ramification points of the algebra $\mathcal{A}$ correspond to some divisors of the polynomial $G(x_1,\dots,x_m)$. Then by Theorem 11 the algebra $\mathcal{A}$ is Faddeev equivalent over $k(x_1,\dots,x_{m-1})$ to an algebra $\mathcal{B}$ of index at most $n^{[(d_m+1)/2} = n^{M_m}$.

Suppose that $\mathcal{A} \sim \otimes_i (U_i, V_i)_n$, where the polynomials $U_i, V_i \in k[x_1,\dots,x_m]$ are irreducible and $\mathrm{GCD}(U_i, V_i) = 1$. Let $H_i,\dots,H_q$ be all the irreducible pairwise coprime polynomials occurring in the symbol-algebras in the decomposition of $\mathcal{A}$. By Lemma 14 there exists $a \in k$ such that the polynomials $\overline{H}_i$ are coprime. Then $\mathcal{A}_{x_m-a} \sim \otimes_i(\overline{U}_i, \overline{V}_i)_n$ and $\mathrm{GCD}(\overline{U}_i, \overline{V}_i) = 1$. We claim that the absence of ramification of $\mathcal{A}$ at $H \in \{H_1,\dots,H_l\}$ implies the absence of ramification of $\mathcal{A}_{x_m-a}$ at $\overline{H}$. We rewrite $\mathcal{A}$ in the form

$$\mathcal{A} \sim (H, U)_n \otimes (\otimes_i (S_i, T_i)_n),$$

where $\mathrm{GCD}(H, U) = \mathrm{GCD}(H, S_i) = \mathrm{GCD}(H, T_i) = 1$. Then

$$\mathcal{A}_{x_m-a} \sim (\overline{H}, \overline{U})_n \otimes (\otimes_i (\overline{S}_i, \overline{T}_i)_n),$$

and $\mathrm{GCD}(\overline{H}, \overline{U}) = \mathrm{GCD}(\overline{H}, \overline{S}_i) = \mathrm{GCD}(\overline{H}, \overline{T}_i) = 1$ by the choice of a. Since $\mathcal{A}$ is unramified at H, we have $U \in (k(X_H)^*)^n$ where the set $X_H \subset \mathrm{A}^m_k$ is defined by the equation $H = 0$. Then there exist $G_1, G_2 \in k[x_1,\dots,x_m]$ such that $U = G_1^n + G_2 H$, which implies that $\overline{U} = \overline{G}_1^n + \overline{G}_2\overline{H}$. We consider the ramification of the algebra $\mathcal{A}_{x_m-a}$ at some irreducible divisor h_0 of the polynomial H. The ramification at h_0 is defined by the element $\overline{U} \in k(Y_{h_0})$ where $Y_{h_0} \subset \mathrm{A}^m_k$ is defined by the equation $h_0 = 0$. But, since $\overline{U} = \overline{G}_1^n + \overline{G}_2\overline{H}$ and h_0 divides $\overline{H}$, we have $\overline{U} \in k(Y^*_{h_0})^n$, that is, there is no ramification at h_0. Thus, if $\mathcal{A}$ is unramified at H, then $\mathcal{A}_{x_m-a}$ is unramified at the divisors of $\overline{H}$. Consequently, all the finite ramification of the algebra $\mathcal{A}_{x_m-a}$ occurs only at some divisors of the polynomial $\overline{F}_i$.

We have,

$$\mathcal{B}_{x_m-a} = \mathcal{B} \otimes k(x_1,\dots,x_{m-1})(x_m)_{x_m-a} \sim \mathcal{C} \otimes_{k(x_1,\dots,x_{m-1})} k(x_1,\dots,x_{m-1})(x_m)_{x_m-a},$$

where $\mathcal{C}$ is some c.s.a. over $k(x_1,\dots,x_{m-1})$. Then

$$i(\mathcal{C} \otimes_{k(x_1,\dots,x_{m-1})} k(x_1,\dots,x_{m-1})(x_m)_{x_m-a}) \le i(B) \le n^{M_n}$$

and

$$(\mathcal{B} \otimes (\mathcal{C}^{-1} \otimes k(x_1,\dots,x_m)))_{x_m-a} \sim 1.$$

We consider the algebra $\mathcal{A}_{x_m-a}$ which satisfies

$$\mathcal{A}_{x_m-a} \sim \mathcal{D} \otimes k(x_1,\dots,x_{m-1})(x_m)_{x_m-a},$$

where $\mathcal{D}$ is a c.s.a. over $k(x_1,\dots,x_{m-1})$. Lemma 14 implies that the algebra $\mathcal{D}$ can be ramified only at some irreducible divisor of the polynomial $\overline{G}$. We observe that the degree of the polynomial $\overline{G}$ in the variable x_j is at most d_j. Consequently, by the induction hypothesis we have

$$i(\mathcal{D}) \le n^{r+M_1+\dots+M_{s-1}+min_{1\le i\le m-1}\{\sum_{1=j\ne i}^{m-1} M_j\}}.$$

Further, $\mathcal{A}\otimes(\mathcal{D}^{-1}\otimes k(x_1,\dots,x_m))_{x_m-a}\sim 1$. Consequently, the algebras $\mathcal{A}\otimes(\mathcal{D}^{-1}\otimes k(x_1,\dots,x_m))$ and $\mathcal{B}\otimes(\mathcal{C}^{-1}\otimes k(x_1,\dots,x_m))$ have a trivial completion at a $k(x_1,\dots,x_{m-1}))$-rational point. Thus,

$$\mathcal{A}\sim\mathcal{B}\otimes(\mathcal{D}^{-1}\otimes\mathcal{C}^{-1}\otimes L).$$

Then

$$i(\mathcal{A})\le n^{r+M_1+\dots+M_m+min_{1\le i\le m}\{\sum_{1=j\ne i}^{m-1}M_j\}}.$$

We observe that $\mathcal{A}$ can be regarded as a c.s.a. over the function field of the projective line $\mathbb{P}^1_{k(x_1,\dots,x_{l-1},x_{l+1},\dots,x_m)}$. Then we have the following bound for the index:

$$i(\mathcal{A})\le n^{r+M_1+\dots+M_m+min_{1\le i\le m,i\ne l}\{\sum_{1=j\ne i}^{m-1}M_j\}}.$$

Hence we obtain the following bound:

$$i(\mathcal{A})\le n^{r+M_1+\dots+M_m+min_l\{min_{1\le i\le m,i\ne l}\{\sum_{1=j\ne i}^{m-1}M_j\}\}}=$$

$$n^{r+M_1+\dots+M_m+min_{1\le i\le m}\sum_{1=j\ne i}^{m-1}M_j\}\}}.$$

The theorem is proved.

§5. The case of a projective curve

In this section we consider the case of an arbitrary smooth curve C over k with function field $k(C)$. The results obtained here are based on the following theorem of van den Bergh.

Theorem 15 *[8] Let C be a smooth projective curve over k with a k-rational point P and $\mathcal{D}$ is unramified algebra over $k(C)$ such that $\mathcal{D}\otimes k(C)_P$ is the full matrix algebra over $k(C)_P$, where $k(C)_P$ is the completion of the field $k(C)$ relative to the valuation corresponding to the point P. Then $i(\mathcal{D})$ divides $e(\mathcal{D})^{2g(C)}$, where $g(C)$ is the genus of the curve C.*

The first main result is contained in the following theorem.

Theorem 16 *Suppose that k and all the finite extensions of k are $P_{n,r}$-fields, k contains a primitive nth root of 1, C is a smooth projective curve over k, $\mathcal{A}$ is a central simple algebra over $k(C)$ of exponent $n=p_1^{n_1}...p_q^{n_q}$ ramified at closed points $P_1,...,P_s$ of the curve C, $L=k(P_1,...,P_s)$ is the field of definition of these points, and $t=p_1^{m_1}...p_q^{m_q}u$ is the degree of the extension L over k, where $GCD(n,u)=1$. Then*

$$i(A)\le n^{r+N+2g(C)}p_1^{m_1}...p_q^{m_q},\qquad(4)$$

where N is the number of points of the field $\overline{k}(C)$ over the points P_i.

Proof. The algebra $\mathcal{A}\otimes_{k(C)}L(C)$ has ramification only at N L-rational points. The ramification at these points is determined by elements $a_1,\dots,a_N$ of the field L^*. Then the algebra $\mathcal{B}=\mathcal{A}\otimes L(\sqrt[n]{a_1},\dots,\sqrt[n]{a_N})$ is unramified. Consequently,

$$i(\mathcal{B}\otimes\mathcal{B}_P^{n-1})\le n^{2g(C)}$$

by van den Bergh's theorem, where

$$\mathcal{B}_P = \mathcal{B} \otimes L(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_N})(C)_P,$$

and P is some $L(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_N})$-rational point. Since $L(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_N})$ is a $P_{n,r}$-field, we have $i(\mathcal{B}) \leq n^{r+2g(C)}$. By extending the algebra $\mathcal{A}$ to an extension of degree at most $tn^N = p_1^{m_1}, \ldots, p_q^{m_q} un^N$ we obtain an algebra of index at most $n^{r+2g(C)}$. Hence, $i(A) \leq n^{r+N+2g(C)} p_1^{m_1} ... p_q^{m_q}$.

The second result is based on the following construction. Suppose that k is a $P_{n,r}$-field, C is a plane projective curve over k defined by the equation $F(x, y) = 0$ with k-rational point P, and $\mathcal{A}$ is a c.s.a. over $K(C)$ of exponent $n = p_1^{n_1} ... p_q^{n_q}$. Suppose that $t = p_1^{m_1} ... p_q^{m_q} u$, where $\mathrm{GCD}(n, u) = 1$, is the minimal of the degrees of the polynomial $F(x, y)$ in the variables x and y and that $\mathcal{A}$ is ramified at closed points $P_1, ..., P_s$ of the curve C. Let $Q_1, ..., Q_m$ be distinct closed points of the projective line $\mathbb{P}^1_k$ lying under the points P_i, and let g be the product of all the irreducible unitary polynomials corresponding to finite points lying under the points P_i.

If all the points Q_i are finite, then we set $u = g(P)^{tn-1}$ in the case $g(P) \neq 0$, and $u = 1$ in the case $g(P) = 0$, and set $H = gu$.

If all the points Q_i are finite and $\mathrm{GCD}(deg\ g, n) = 1$, then we set $H = gu$, where u is defined as above. If $\mathrm{GCD}(deg\ g, n) \neq 1$, then let v be a linear polynomial such that $\mathrm{GCD}(v, g) = 1$ and let m be the least positive integer such that $\mathrm{GCD}(deg\ g + m, n) = 1$. Then we set $H = gv^m u$, where

$$u = \begin{cases} (g(P)v^m(P))^{tn-1} & \text{if } g(P)v^m(P) \neq 0; \\ 1 & \text{if } g(P)v^m(P) = 0. \end{cases}$$

Let C_1 be a curve with function field $k(C_1) = k(C)(\sqrt[nt]{H})$. Assuming the notation introduced above we state the following theorem

Theorem 17 *We have*

$$i(\mathcal{A}) = n^{2g(C_1)+r+1}. \tag{5}$$

Proof. The algebra $\mathcal{A} \otimes k(C_1)$ is unramified. In addition, by the construction of the polynomial H the curve C_1 has a k-rational point. Then $i(\mathcal{A} \otimes k(C_1)) \leq n^{r+2g(C_1)}$ by van den Bergh's theorem, which yields $i(\mathcal{A}) \leq n^{r+1+2g(C_1)} p_1^{m_1}, \ldots, p_q^{m_q}$. The theorem is proved.

Remark. Note that in the case of special curves and exponents the bound (5) is sharper than (4).

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