

PORTFOLIO OPTIMIZATION PROBLEMS: A SURVEY

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Abstract

Optimization models play an increasingly role in financial decisions. This paper analyzes the portfolio optimization model which is the most important of them. We are discussing the mathematical models and modern optimization techniques for some classes of portfolio optimization problems more important criteria. Portfolio optimization problems are based on mean-variance models for returns and for risk-neutral density estimation. The mathematical portfolio optimization problems are the quadratic or linear parametrical programming sometimes with integer variables.

Key words: Markowitz, portfolio optimization, absolute deviation, portfolio diversification, efficient frontier, Sharpe ratio, minimax model, integer variables, fuzzy expected return

MSC

1 Introduction

Conception of an optimal portfolio of assets was first time mentioned by Louis Bacheliers in his doctoral thesis which was defended in 1900 in Paris. Unfortunately, this thesis exactly like the theory of optimization created by L. Kantorovich and T. Kupmans the Nobel Prize winners in economy were less common among financial managers. They managed to use primary skills of actuarial mathematics, elementary concepts of share fare value (price). The modern portfolio theory was firstly reviewed in the work written by Markowitz (1950) and Sharpe (1964) who were awarded Nobel Prize in Economics in 1990. This theory is seems to be of high importance. If you make an inquiry about “portfolio theory” and “portfolio optimization” using the search engine Google.com you will be given about 13, 5 mln links for the first one and about 2, 2 mln links for the second one. Moreover the term “portfolio management” has about 13 mln links.

2 The standard Markowitz portfolio model (model based on Euclidean metric of risk estimation)

Let's suppose that investor has the possibility to choose from the variety of different financial assets like securities, bonds and investment projects. The main point is to define investment portfolio $x = (x_1, \dots, x_n)$ where x_j is proportion of the asset j . Then the budget constraint is

$$\sum_{j=1}^n x_j = 1, \quad x_j \geq 0, \quad j = 1, \dots, n. \quad (1)$$

It is valuable to say that absolute weightings of assets could be included in the Markowitz. For instance, by K we denote the investor's initial capital. Then the budget constraint (1) might be replaced for:

$$\sum_{j=1}^n K_j \bar{x}_j = K, \quad x_j \geq 0, \quad j = 1, \dots, n. \quad (1.1)$$

where K_j is the price of asset j . If all assets are infinitely divisible replaced variables $x_j = K_j \bar{x}_j / K$ we get budget constraint (1).

Markowitz portfolio model assumes to use two criterions: portfolio expected return and portfolio volatility (measure of risk adjusted). Important to add that theory uses the historical parameter, volatility, as a proxy for risk, while return is an expectation on the future.

The return $R(x)$ of the portfolio x is the component- weighted expected the return R_j of the constituent assets. The expected return of an asset is a probability-weighted average of the return in all scenarios. Calling p_t the probability of scenario t and r_{ij} the return in scenario t , we may write the expected return as

$$r_j = E \left[R_j \right] = \sum_{t=1}^T p_t r_{ij}$$

It's assumed that all scenario t (historical) are equiprobable in the future then $p_t = 1/T$ and $r_j = \sum_{t=1}^T r_{jt} / T$ (see example 1).

The function of the expected return of the portfolio is needed to be maximized:

$$r \leftarrow E \left(\leftarrow \right) = \sum_{j=1}^n x_j r_j \Rightarrow \max . \quad (2)$$

If we suppose that $r_1 \geq \dots \geq r_n$ then optimal solution of the problem (1), (2) is $x^g = \left(\dots, 0 \right)$, i.e. all capital should invest in the most profitable asset (greedy solution). Clearly, it is very risky. That is why investors add $x_j \leq u_j, j = 1, \dots, n$ (upper bound constraint) to budget constraints. In this case greedy solution has following form

$$x^g = \left(u_1, \dots, u_k, \left(1 - \sum_{j=1}^k u_j \right), 0, \dots, 0 \right),$$

where $\sum_{j=1}^k u_j \leq 1$ and $\sum_{j=1}^{k+1} u_j \geq 1$ and stays optimal. It is possible further to add constraints for diversification of risks (see chapter 14). However, Markowitz proposed other approach.

One of the best-known measures of risk is standard deviation of expected returns. Let's σ_{ij} is covariance of the returns i and j , i.e. $\sigma_{ij} = \frac{1}{T} \sum_{t=1}^T \left(r_{it} - r_j \right) \left(r_{jt} - r_j \right)$. Markowitz derived the general formula for the standard deviation of the portfolio as follows:

$$\sigma(x) = \sqrt{E \left[\left(R(x) - r(x) \right)^2 \right]} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n x_i \sigma_{ij} x_j} \Rightarrow \min . \quad (3)$$

The variance of all asset's returns is the expected value of the squared deviations from the expected return:

$$\sigma^2 = \sum_{t=1}^T p_t \left(\leftarrow - E \left(\leftarrow \right) \right)^2$$

Remark that the covariance matrix $\sigma = [\sigma_{ij}]_{n \times n}$ is positively semi-definite and consequently $\sigma(x)$ and $\sigma^2(x)$ are convex functions. That is why standard Markowitz portfolio model (1) – (3) is bicriteria optimization problem with linear (2) and convex quadratic (3) objective functions.

In some occasions standard deviation could be substituted for k -order target risk:

$$\sigma(x) = E \left[\left(R(x) - R(x) \right)^k \right]^{1/k} .$$

A portfolio x is **efficient/ Pareto optimal** if and only if no other feasible portfolio that improves at least one of the two optimization criteria without worsening the other. An efficient portfolio is the portfolio of risky assets that gives the lowest variance of return of all portfolios having the same expected return. Alter-

natively we may say that an efficient portfolio has the highest expected return of all portfolios having the same variance. The **efficient frontier** sur plane (r, σ) is the image $(r(x), \sigma(x))$ of all efficient portfolios x .

While choosing an efficient portfolio we could apply for weighting objective function approach. The approach is based on using the Carlin theory of coincidence Pareto-optimal solutions in (1) – (3) in optimal solutions in the one-criterion parametric optimization with parameter λ :

$$\lambda r(x) - (1 - \lambda)\sigma(x) \rightarrow \max \quad (4)$$

Here the parameter $\lambda(0 \leq \lambda \leq 1)$ shows investor's risk. The lower $\lambda=0$ the less risk we apply for the model, investor is more conservative. If $\lambda=1$ investor must accept risk in order to receive higher returns.

This algorithm for parametric quadratic programming solves the problem (1)-(4) for all λ in the interval $[0, 1]$. Starting from one point on the efficient portfolio the algorithm computes a sequence of so called **corner portfolios** $\{x^1, \dots, x^m\}$. These corner portfolios define all efficient portfolio are convex combinations of the two adjacent corner portfolios: if x' and x'' are adjacent corner portfolios with expected returns $r(x')$ and $r(x'')$, $r(x') \leq r(x'')$ then for every $r(x^\mu) = \mu r(x') + (1-\mu)r(x'')$ the efficient portfolio x^μ is calculated as $x^\mu = \mu x' + (1-\mu)x''$.

The second approach leads to the task of minimizing the variance of the portfolio return given a lower bound on the expected portfolio return

$$r(x) \geq \lambda, \quad (4.1)$$

i.e. under all possible portfolios x , consider only those which satisfy the constraints, in particular those which return at least an expected return of λ . Then among those portfolios determine the one with the smallest return variance. Problem (1), (3), (4.1) is quadratic optimization problem with a positive semi-definite objective matrix σ . This problem can be solved in a very efficient way by using standard quadratic programming algorithms.

The third approach we consider the task of maximizing the mean of the portfolio return $R(x)$ under a given upper bound λ for the variance $\sigma(x)$:

$$\sigma(x) \leq \lambda. \quad (4.2)$$

Problem (1), (2), (4.2) is a linear parametric programming with an additional convex quadratic constraint (4.2) and parameter λ .

3 Model with risk-free asset (Tobin model)

Risk-free asset hypothetically corresponds to be short-term government securities. Conditionally it is assumed that the variation of the government securities return r_0 is equal zero. Considering the following model for portfolio $x = (x_0, x)$ with risk free asset x_0 :

$$x_0 + \sum_{j=1}^n x_j = 1, \quad x_j \geq 0, \quad j = 0, \dots, n \quad (1.2)$$

$$r(x) = r_0 x_0 + \sum_{j=1}^n r_j x_j \rightarrow \max \quad (2.1)$$

$$\sigma = \sqrt{x_0^2 \sigma_0^2 + x_p^2 \sigma_p^2 + 2x_0 x_p \sigma_{0p}} = \sqrt{x_0^2 \sigma_0^2 + x_p^2 \sigma_p^2 + 2x_0 x_p \sigma_{0p}} = \sqrt{x_p^2 \sigma_p^2} = x_p \sigma_p = \sigma(x) \rightarrow \min \quad (3.1)$$

Obviously, the expected rates of return on all risky assets are not less asset, i.e. $r_j \geq r_0$.

If we take some definite efficient portfolio we could figure all portfolios with risk free asset x_0 on CML (Capital Market Line):

$$E(R_c) = r_0 + \sigma_c \frac{E(r_m) - r_0}{\sigma_m},$$

where r_m is return of the market portfolio (depending on the market index and its risk is σ_m).

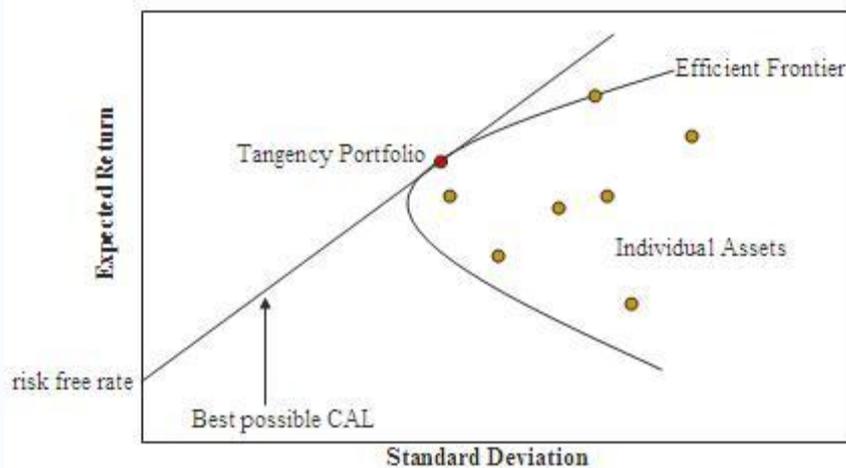


Fig 1. The efficient frontier

Example 1

We apply Markowitz's model to the problem of the optimization portfolio of blue chips, Hi-Tech corporation's shares, real estate and Treasure bonds. The annual times series for the return are given below for each asset between six years.

Average annual percentage

j/t		r_{jt}						$r_j = E(R_j)$
		1	2	3	4	5	6	
Blue chips	x_1	18,24	12,12	15,23	5,26	2,62	10,42	10,6483
Hi-Tech shares	x_2	12,24	19,16	35,07	23,46	-10,62	-7,43	11,98
Real estate Market	x_3	8,23	8,96	8,35	9,16	8,05	7,29	8,34
Treasury Bonds	x_4	8,12	8,26	8,34	9,01	9,11	8,95	8,6317

Average annual percentage r_{jt} is specified

$$r_{jt} = \frac{P_{jt+1} - P_{jt}}{P_{jt}},$$

where P_{jt} is asset price j at instant time t . Then covariance matrix is:

$$\sigma = \begin{pmatrix} 29.0552 & 40.3909 & -0.2879 & -1.9532 \\ 40.3909 & 267.344 & 6.8337 & -3.6970 \\ -0.2879 & 6.8337 & 0.3759 & -0.0566 \\ -1.9532 & -3.6970 & -0.0566 & 0.1597 \end{pmatrix};$$

It is interesting to compute the volatility of the return on each asset $\sigma_i = \sqrt{\sigma_{ii}}$ and the correlation matrix:

$$\begin{aligned} \sigma^2 \left(\sum_{j=1}^4 x_j \right) &= 29.0552x_1^2 + 80.7818x_2x_1 - 0.5758x_3x_1 - 3.9064x_4x_1 + 267.344x_2^2 + \\ &+ 0.3759x_3^2 + 0.1597x_4^2 + 13.6673x_2x_3 + 7.3940x_2x_4 - 0.1133x_3x_4 \rightarrow \min \\ x_1 + \dots + x_4 &= 1, \quad x_j \geq 0, \quad j=1, \dots, 4 \\ 10.6483x_1 + 11.98x_2 + 8.34x_3 + 8.6317x_4 &\geq \lambda \end{aligned}$$

$$p_{ij} = \frac{\sigma_{ij}}{\sigma_i\sigma_j}$$

Setting in the problem (1), (3), (4.1) for portfolio optimization and solving it for par example $\lambda=10\%$. We get the optimal portfolio ($x_1 = 0.6806, x_2 = 0, x_3 = 0.0148, x_4 = 0.3046$) with risk $\sigma \left(\sum_{j=1}^4 x_j \right) = \sqrt{x\sigma x} = 3.5577\%$. Minimal risk is 0.0884 with portfolio ($x_1 = 0.5370, x_2 = 0, x_3 = 0.1777, x_4 = 0.7687$) and return 8.687%.

4 Multi-objective model for portfolio optimization

The main problem in optimization portfolios is that the portfolios are extremely concentrated on a few assets which are a contradiction to the notion of diversification. Therefore there is scope for introducing another criterion with one for diversification and the best candidate for this. They usually solve quadratic problem for portfolio optimization and then apply entropy measure for infer how much portfolio is diversified. In paper [Jana, Roy, Mazumber (2007)] supplement maximize Shannon's entropy and skewness of portfolio

$$E_n \left(\sum_{j=1}^n x_j \log x_j \right) \rightarrow \max \quad (5)$$

$$S \left(\sum_i \sum_j \sum_k \gamma_{ijk} x_i x_j x_k \right) \rightarrow \max \quad (6)$$

where $\gamma_{ijk} = E \left[(R_i - r_i)(R_j - r_j)(R_k - r_k) \right]$ is central third moment of returns.

5 Model based on Minkowski absolute metric of risk estimation

Konno and Yamazaki (1991) propose a linear programming model instead of the quadratic model. Quite widespread to evaluate risk using the Minkowski metric l_1 in which deviation is sum of absolute values, i.e. risk l_1 of the portfolio return (absolute deviation) is defined as

$$\sigma(x) = E \left[|R - E(R)| \right] = E \left[\left| \sum_{j=1}^n r_j x_j - E \left[\sum_{j=1}^n r_j x_j \right] \right| \right]$$

Under the assumption on normal distribution the absolute deviation is equivalent to the standard deviation as the measure of risk [Konno and Yamazaki (1991)].

That allow insert additional variables y_t into the model (1), (2) and

$$\sum_{t=1}^T y_t \rightarrow \min, \quad (3.1)$$

under the condition

$$y_t + \sum_{j=1}^n (r_{jt} - r_j) x_j \geq 0, \quad t = 1, \dots, T \quad (3.2)$$

$$y_t - \sum_{j=1}^n (r_{jt} - r_j) x_j \geq 0, \quad t = 1, \dots, T. \quad (3.3)$$

Remark that variable y_t may take either sign. In this model it is only necessary to solve a linear optimization problem.

According to Konno and Yamazaki the mean absolute deviation portfolio optimization model's advantages over the Markowitz's model are (i) this model does not use the covariance matrix which therefore does not need to be calculated, (ii) solving this linear model is much easier than solving a quadratic model.

In doing so it is possible to differentially penalize the upside from the downside deviation of the portfolio return from its mean. Let p_u and p_d denote penalty parameters for the upside and downside errors respectively. Then constrains (3.2) and (3.3) replaced

$$y_t + p_d \sum_{j=1}^n (r_{jt} - r_j) x_j \geq 0, \quad t = 1, \dots, T \quad (3.2')$$

$$y_t - p_u \sum_{j=1}^n (r_{jt} - r_j) x_j \geq 0, \quad t = 1, \dots, T \quad (3.3')$$

For the old model use a symmetric penalty with $p_u = p_d = 1$. Of particular interest is the case where $p_u = 0$ and hence the model will penalize only downside risk.

Feinstein and Thapa (1993) modified model (3.1)-(3.3) proposed a following model that is equivalent to Konno and Yamazaki's

$$\min \sum_{t=1}^T u_t + v_t$$

subject to (1), (2) and

$$u_t + v_t - \sum_j (r_{jt} - r_j) x_j = 0, t=1, \dots, T, \quad (3.2'')$$

$$u_t, v_t \geq 0 \quad (3.3'')$$

6 Model based on Minkowski semi-absolute metric of risk estimation

In the standard Markowitz portfolio model risk is estimated by the standard deviation with Euclidean metric. Also applicable to use lower semi-variation in order to estimate a portfolio risk:

$$\sigma_-(x) = \sqrt{E \left[\max\{0, -a\} \right]^2},$$

where $a = \max\{0, -a\}$ losses of expected return are taken into account.

Extension of the semi-variance measure only computed expected return below zero (that is negative returns) or returns below some specific asset such as T-bills, the rate of inflation or a benchmark. These measures of risk implicitly assume that investors want to minimize the damage from returns less than some target risk.

The main point of the model is to find an optimal portfolio in order to minimize sum of out of condition losses [Speranza (1993)]. Therefore the risk is estimated by semi-absolute deviation:

$$\sigma(x) = \frac{1}{T} \sum_{t=1}^T \left| \min \left\{ 0, \sum_{j=1}^n (r_{jt} - r_j) x_j \right\} \right|.$$

Let's assume while choosing the portfolio that if in the history repeats itself then losses will be minimal. The given model is the module of the cautious investor. Certainly it is not applicable if future tendency is fundamentally different from historical trends.

Let's insert variables y_t each of which represents losses of a portfolio x in the period of time t . Then portfolio optimization problem with semi-absolute deviation can be defined:

$$\min \sum_{t=1}^T y_t,$$

on conditions that(1), (2) and

$$y_t + \sum (r_{jt} - r_j)x_j \geq 0, \quad y_t \geq 0, \quad t = 1 \dots T.$$

Since the model based on a mean semi-absolute deviation risk is bicriterial linear programming model with a smaller number of constraints.

7 Model based on Chebyshev metric of risk estimation (Maxmin and Minimax model)

Young (1998) introduced a minimax portfolio optimization criterion which defines the optimal portfolio as that one that would maximize minimum the return over all the past historical periods. Risk of the portfolio x in this model stands as the measure during the most unsuccessful worst case periods of historical trends, i.e. in metric l_∞ :

$$r \leftarrow \min_{t=1 \dots T} \sum_{j=1}^n r_{jt} x_j \rightarrow \max$$

According to this criterion (3) and budget constraints (1) which is system of linear inequality with parameter λ :

$$\sum_{j=1}^n r_{jt} x_j \geq \lambda, \quad t = 1, \dots, T,$$

and objective function $\max \lambda$ we get simple bicriterial linear programming problem

It is worth nothing that Papahristodoulou and Dotzauer (2004) compared Markowitz's model and Young's model.

Cai et al (2000) proposed an alternative minimax risk function in portfolio optimization. The super cautious investor always tries to combine his portfolio proposing that if historical (scenario) situation repeats he should get highest possible earnings from portfolio (losses are minimal in case $R(x)$ is negative value).

Such a risk function is defined as the average of maximum individual risks over number of past time periods, using the maximum absolute deviation risk model l_∞ (Cai's model)

$$\max_{j=1, \dots, n} E |R_j x_j - E R_j x_j| \rightarrow \min.$$

The alternative l_∞ risk function is defined as (Teo's model, see [Teo (2001)]):

$$\frac{1}{T} \sum_{t=1}^T \max_{j=1, \dots, n} E |R_{jt} x_j - r_{jt} x_j| \rightarrow \min.$$

These models can be transformed into the following linear forms (1),(2) and

$$y \rightarrow \min$$

$$E |R_j - r_j| x_j \leq y, \quad j = 1, \dots, n$$

or

$$\sum_{t=1}^T y_t \rightarrow \min$$

$$E |R_{jt} - E R_{jt}| x_j \leq y_t, \quad t = 1, \dots, T, \quad j = 1, \dots, n.$$

8 Sharpe model with fractional criteria

The main content of this model is replacement of the bicriterion model (1), (2), (3) for the one-criterion model and budget constraint (1) with linear-fractional objective function [(see Sharpe (1994))]:

$$\sum_{j=1}^n r_j x_j / \sqrt{\sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij}} \rightarrow \max . \quad (\text{Sharpe-ratio})$$

In this form, this problem is not easy to solve. In [Cornujeols, Tutuncu (2007)] describes a direct method to obtain the optimal risky portfolio by constructing a convex quadratic programming problem equivalent to (Sharpe-ratio).

9 Linear models of returns

These models are based on the Sharpe idea to present expected return function of the market coefficients (market index, GDP, inflation index and etc). Let it be R_m is the return for the aggregate stock market (market index). More particularly to use single-factor model:

$$r_j = \alpha_j + \beta_j R_m + \varepsilon_j,$$

in which assets return r_j is the sum of: linear function $\beta_j R_m$ with coefficient β_j (beta- coefficient), which shows share sensitivity asset j to market trend, constant α_j of the asset j (alpha-coefficient) which doesn't depend on the market conditions and random variable ε_j with $E(\varepsilon_j) = 0$. It's supposed that ε_j and R_m are independent, i.e. its covariation is equal zero. In compliance with made assumption expected return of the portfolio x is equal:

$$R(x) = \sum_{j=1}^n x_j (\alpha_j + \beta_j E(R_m) + \varepsilon_j),$$

and it's risk

$$\sigma(x) = \sum_{j=1}^n x_j \beta_j^2 \sigma_m^2 + \sum_{j=1}^n x_j^2 \sigma_{\varepsilon_j}^2 + \sum_{j \neq i} x_i x_j \beta_i \beta_j \sigma_m^2.$$

Other more simply criteria firstly assumed by W. Sharpe [see Sharpe (1963)]

$$\sum_{j=1}^n r_j x_j / \sum_{j=1}^n \beta_j x_j \rightarrow \max ,$$

where β_j is regression coefficient between dividend assets j and market index.

Let's give the example of similar function

$$r_j = 0.045 + 0.06\beta_j, \quad \beta_j = \frac{\sigma_j \rho_{jDAX}}{\sigma_{DAX}},$$

where ρ_{jDAX} is correlation between asset j and index DAX (30 benchmark German companies) of the historical data.

10 Model with limited number assets (cardinality constrained)

Generally investors incline to limit number of assets m included in the portfolio. Markowitz model with additional discrete (boolean) variables δ_j include the following meaning: $\delta_j=1$ - asset j is **putted** on the portfolio, $\delta_j=0$ - asset j is not putted on. Then new constraints are following (a small number of assets):

$$\sum_{j=1}^n \delta_j \leq m, \quad \delta_j = 0 \text{ or } 1, \quad x_j \leq \delta_j, \quad j = 1, \dots, n,$$

and new model of portfolio optimization is mixed integer programming problem.

11 Model with buy-in thresholds

Buy-in thresholds prevent assets from being included in a portfolio with small weights only. They determine that asset weights are either above a lower bound l_j or the asset is not part of the portfolio at all. The main reason for such a constraint might be that some costs are – at least partially- determined by the number of different asset (shares) that are held (e.g. information costs, fixed transaction costs). Jobst et al (2001) have shown that a portfolio optimization problem with buy-in thresholds can be formulated as a mixed- integer programming (1)-(3) and supplement constraint.

$$l_j \delta_j \leq x_j, \quad j=1, \dots, n \quad (\text{thresholds constraint})$$

For example, it's common for German Investment Law to use constraint (5, 10, 40). The point of this rule is that investor should combine no more than 40% of mutual funds shares in portfolio, less than 10% certain type shares in the portfolio and shares of the same issuer are allowed to amount to up to 5%. These conditions could be modeled by following limits:

$$\sum_{j=1}^n x_j \delta_j \leq 0.4, \quad x_j - 0.05 \delta_j \leq 0.05, \quad j = 1, \dots, n. \quad (5,4,10\text{-constraint})$$

12 Models with transactions costs

In the Markowitz's classical work transaction costs associated with buying and selling of equities were not allowed. The objective is to find the portfolio x that has minimal transactions costs.

Let's bring to the return model transaction costs $d_j x_j$ of the acquiring asset j . Thus the function of return takes a form:

$$\sum_{j=1}^n (r_j - d_j) x_j \rightarrow \max \quad (2.1)$$

Inserted variable do not changed an essence of the objective function. Some of the economists give considerations towards the concave function of the transaction costs $d_j(x_j)$. In this criterion (2.1) becomes convex.

It is supposed to be more complicated to create a model of fixed costs f_j which do not depend on the size of acquiring assets, f_j is a payment for market entering j . The fixed costs are discrete and it's assumed the inserting of Boolean variables δ_j . The criterion of expected return (2) in this case is replaced on:

$$\sum_{j=1}^n (r_j x_j - f_j \delta_j) \rightarrow \max \quad (2.1')$$

and it adds constraint

$$x_j \leq \delta_j, \delta_j = 0 \text{ or } 1 \quad j = 1, \dots, n.$$

13 Model with integer (lot) assets

It is supposed under the Markowitz model that investment capital and its equal 1 and portfolio x combine shares of the assets. At some times shares of the assets could be multiple of the asset value. For instance, at the moment of purchasing asset j has actual price p_j or asset j sells by lots in quantity $p_j, 2 p_j, 3 p_j, \dots$

According to this let insert new variable y_j , which indicate quantity of the asset j , to be included in a portfolio should be an integer multiple of the number of lot, usually 1000 stocks in the Tokyo Stock Exchange. Well the equation (1) should be substituted for inequality:

$$K_0 \leq \sum_{j=1}^n p_j y_j \leq K_1, \quad x_j \geq 0 \text{ and integer}, \quad j = 1, \dots, n \quad (1.1')$$

where K_0, K_1 are upper and lower limit of the investor's capital. The integer variable y_j represents the number of lots for each asset j which will make part of the optimal portfolio.

Mansini and Speranza (1999) present three different heuristics for model (1.1) with integer variables (using data from the Milan Stock Exchange). The heuristics proposed are based the idea of constructing and solving mixed integer subproblems with consider subsets. The subsets are generated by exploiting the information obtained from the relaxed linear optimization problem.

Integer variables, sometimes also called minimum transaction lots or round lots, are another type of "complex" constrain often mentioned in publications [see e.g. Mansini and Speranza (1999)], Lin and Wang (2002), Maringer (2002), Li et al (2006)]. Another way to handle this would be to introduce an asset that represents cash and is divisible up to the smallest currency unit (e.g. cent, ounce of gold). The problem (1.1), (2), (3) becomes considerably more mathematically compound and concerns to category of the integer quadratic programming.

14 Model with submodular constraints of diversification of risks

Owing to additional linear constraints the diversification of risks in the models is reached. It's possible to built a model in which these constrains will have simple structure, i.e. feasible set is polymatroid. Most of the restrictions of investor can be expressed as simple constraints:

lower and upper bounds on individual assets, section and regional constrains, i.e. constraints are invested in certain sectors financial assets (e.g. in

shares of the energy sector and at least 60% of the budget has to be invested in European shares):

$$l(I) \leq \sum_{j \in I} x_j \leq u(I), \quad (\text{submodular diversification})$$

where I is the set of those assets that belong to the restricted sector (region and etc.). Maximal number of these constraints can amount 2^n . Clearly, that when $I=\{j\}$ constraints give lower and upper bounds on individual assets.

If functions $l(I)$ is supermodular, i.e.

$$l(I \cup J) \geq l(I) + l(J) - l(I \cap J)$$

and $u(I)$ is submodular, i.e.

$$u(I \cup J) \leq u(I) + u(J) - u(I \cap J)$$

then budget constraints and diversification constraints give some polymatroid even with additional constraint x_j is integer. In this case, greedy solution is optimal solution in the problem with criterion (2).

15 Model using fuzzy expected return

Choosing optimal portfolios, fuzzy decision theory provides an excellent framework for analysis. Here two reasons: it guaranties a minimum rate of return and gets returns above the risk-free rate for certain market scenarios.

Some authors use fuzzy numbers to represent the future return of assets that approximated as fuzzy numbers the expected return and risk are evaluated by interval-valued means [Dubois and Prade ()]. Let us denote by \tilde{r}_j the fuzzy return on the asset j in the portfolio $P(x)$, then its interval-valued mean is defined as the following interval:

$$E(\tilde{r}_j) = [E_*(\tilde{r}_j), E^*(\tilde{r}_j)]$$

We consider a fuzzy portfolio optimization problem, assuming that the returns assets are modelled by means of a trapezoidal fuzzy number. A fuzzy number \tilde{A} is said to be a trapezoidal fuzzy number $\tilde{A} = (a_l, a_u, c, d)$ if its membership function has the following form (see fig 2).

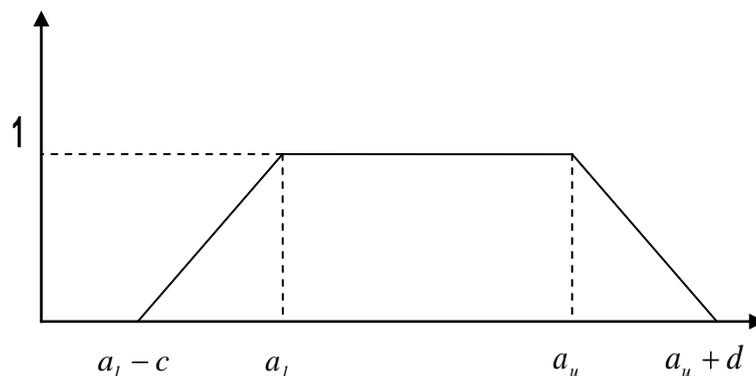


Fig 2. Trapezoidal fuzzy number

If in addition $a_l = a_u$ it is a triangular fuzzy number. On introduce the downside fuzzy risk for a trapezoidal portfolio using the following definition:

$$\sigma(x) = E \left(\max \left\{ E \left(\tilde{r} \right) - \tilde{r}, 0 \right\} \right)$$

This function only penalizes the negative deviations on the expected return. If $\tilde{r}_j = (l_j, a_{uj}, c_j, d_j)$ the trapezoidal return on the j asset, $j=1, \dots, n$ and let $\tilde{R} = (R_l, R_u, C, D)$ the return of the portfolio x , then $\max \left\{ E \left(\tilde{R} \right) - \tilde{R}, 0, R_u - R_l + D(x)/2, 0, C(x) \right\}$.

An essential question connected with solving the fuzzy portfolio optimization problem is related to the defuzzification process for minimization the fuzzy downside for risk considered as a crisp objective and maximize the expected return:

$$\sum_{j=1}^n \left(a_{uj} - a_{lj} + \frac{1}{2} (c_j + d_j) \right) x_j \rightarrow \min$$

(J)

or when the interval-valued possible mean is used, the objective functions are the following:

$$\sum_{j=1}^n \left(a_{uj} - a_{lj} + \frac{1}{3} (c_j + d_j) \right) x_j \rightarrow \min$$

$$\sum_{j=1}^n \left(\frac{1}{2} (a_{uj} + a_{lj}) + \frac{1}{6} (c_j - d_j) \right) x_j \rightarrow \max$$

16 Conclusions

The expected return and the risk measured by the variance are the two main characteristics of an optimal portfolio. The optimal portfolio is desirable (the target portfolio). the real portfolio of assets can not be done by human intuition alone and some other characteristics [Bevtsimas et al (1999)]: (1) closeness to the target portfolio; (2) exposure to different economic sectors close to that of the target portfolio; (3) a small number of names; (4) a small number of transactions; (5) high liquidity; (6) low transaction costs.

The mathematical problem can be formulated in many ways but the principal problems can be summarized as follows:

1. Bicriterial convex quadratic optimization with simple budget constraints
2. Bicriterial linear optimization
3. Linear optimization with simple polymatroidal budget and risk diversification constraints
4. Convex quadratic or linear bicriterial optimization with integer (mixed integer variables)

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