Effective Conductivity of a Composite Material with Non-ideal Contact Conditions

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The effective conductivity of 2D doubly periodic composite materials with circular disjoint inclusions under non-ideal contact conditions on the boundary between material components is found. The obtained explicit formula for the effective conductivity contains all parameters of the considered model such as the conductivities of matrix and inclusions, resistance coefficients, radii and centers of the inclusions, and also the values of special Eisenstein functions. The method of functional equations is used to analyze the conjugation problem for analytic functions which is equivalently derived from the initial problem. Existence and uniqueness for the problem solution is obtained by using a reduction to certain mixed boundary value problem for analytic functions in special functional spaces.

Keywords: 2D composite material; steady-state conductivity problem; effective conductivity; non-ideal contact conditions; functional equations

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1. Introduction

This paper is devoted to the study of analytical solutions of mixed boundary value problems for multiply connected domains and their application to determining a formula for the effective properties of 2D composite materials.

We consider fibrous composite materials with disjoint parallel inclusions which can be geometrically represented as doubly periodic multiply connected domains. The steady-state conductivity problem for such materials is studied. The problem of determining the effective properties of the composites turns out to be equivalent to the problem of finding a potential function satisfying the Laplace equation in each inner point of a considered domain and satisfying certain conditions on the boundary of the domain. Ideal or non-ideal contact conditions can be incorporated as boundary conditions. The derived mixed boundary value problem can be reduced to an equivalent conjugation problem for analytic functions which is solved by the method of functional equations (such method is extensively developed in [12]).

As for other works using the method of functional equations for analogous problems, we point out that a mixed boundary value problem to determining the effective conductivity of unbounded composite materials with infinitely many equal disjoint inclusions in the case of ideal contact conditions is investigated in [2] and [13]. Power series formula for periodic distributions of inclusions under ideal contact conditions are applied in [3] for the study of the effect of polydispersity in the
random distributed inclusions case. In the case of non-ideal contact conditions, a mixed boundary value problem for a bounded region with a finite number of inclusions was solved in [4]. Analytical and numerical study of the effective conductivity problem under non-ideal contact conditions is performed in [5]. Variational methods are applied in the context of non-ideal contact to identify size effects and critical dimensions in [9].

In some cases the homogenization method (cf., e.g. [1], [8]) allow us to reduce the problem for composites with a rich microstructure to a doubly periodic problem (e.g. based on the idea of the so-called representative cell [11]). Doubly periodic composites are investigated in the framework of different physical models. Thus in [14] the homogenization method is used in the study of elasticity problems, diffusion problems, and thermoelasticity problems. The dominated terms of the exact solution are obtained in [14] as solutions to the problem for doubly periodic composites. In [6] elasticity problems for doubly periodic composites are solved by means of the integral equation method.

In this paper, we provide an analytic investigation of the steady-state potential conductivity problem under non-ideal contact conditions on the boundary between the material components. Here, it is considered 2D unbounded doubly periodic composite materials with a large quantity of circular disjoint inclusions of different radii. The conductivity problem is written in the form of a mixed boundary value problem for the Laplace equation in a multiply connected domain. Such a problem is equivalently rewritten as a system of functional-differential equations, and an explicit formula for the effective conductivity is obtained. This formula contains all parameters of the considered model such as the conductivity of the matrix and inclusions, resistance coefficients, radii, centers of the inclusions, and also the values of special Eisenstein functions. The solvability of the problem is performed in certain spaces of analytic functions. The compactness of the corresponding operators plays a central role in the proofs of the results. Conditions are found under which the method of successive approximations is successfully applied to the problem under consideration.

2. Problem formulation and consequent structural relations

In this section we provide a mathematical formulation of the problem in the form of a boundary value problem, and start to analyse some of the consequences of the stated conditions.

In here, it is important to have in mind that according to the theory of homogenization (cf. [1] and [8]) it is sufficient to study properties of a composite material for one of its periodic cells which is called the minimal representative cell [11].

We consider a lattice $L$ which is defined by the two fundamental translation vectors “1” and “i” (where $i^2 = -1$) in the complex plane $C \cong \mathbb{R}^2$ (with the standard notation $z = x + iy$). Here, the representative cell (see Fig. 1) is the square

$$Q_{(0,0)} := \left\{ z = t_1 + it_2 \in \mathbb{C} : -\frac{1}{2} < t_p < \frac{1}{2}, p = 1, 2 \right\}.$$  \hspace{1cm} (1)

Let $E := \bigcup_{m_1, m_2} \{m_1 + im_2\}$ be the set of the lattice points, where $m_1, m_2 \in \mathbb{Z}$. The cells corresponding to the points of the lattice $E$ are denoted by

$$Q_{(m_1, m_2)} = Q_{(0,0)} + m_1 + im_2 := \left\{ z \in \mathbb{C} : z - m_1 - im_2 \in Q_{(0,0)} \right\}.$$ \hspace{1cm} (2)
It is considered the situation when mutually disjoint disks (inclusions) of different radii \( D_k := \{ z \in \mathbb{C} : |z - a_k| < r_k \} \) with boundaries \( T_k := \{ z \in \mathbb{C} : |z - a_k| = r_k \} \) \((k = 1, 2, \ldots, N)\) are located inside the cell \( Q_{(0,0)} \) and periodically repeated in all cells \( Q_{(m_1,m_2)} \). We denote by

\[
D_0 := Q_{(0,0)} \setminus \left( \bigcup_{k=1}^{N} D_k \cup T_k \right)
\]

the connected domain obtained by removing of the inclusions from the cell \( Q_{(0,0)} \); cf. Fig. 1.

Let us consider the problem of determination of the effective conductivity of a double periodic composite material with matrix

\[
D_{\text{matrix}} = \bigcup_{m_1, m_2} \left( (D_0 \cup \partial Q_{(0,0)}) + m_1 + m_2 \right)
\]

and inclusions

\[
D_{\text{inc}} = \bigcup_{m_1, m_2} \bigcup_{k=1}^{N} (D_k + m_1 + m_2)
\]

occupied by materials of conductivities \( \lambda_m > 0 \) and \( \lambda_{i,k} > 0 \), respectively. We restrict our attention to the steady-state case of a potential field in each component of the composite. This problem is equivalent to the determination of the potential of the corresponding fields, i.e., a function \( u \) satisfying the Laplace equation

\[
\Delta u(z) = 0, \quad z \in D_{\text{matrix}} \cup D_{\text{inc}},
\]

which have to satisfy certain boundary conditions on all \( T_k, k = 1, 2, \ldots, N \). We consider these conditions in the following form:

\[
\lambda_m \frac{\partial u^+}{\partial n}(t) = \lambda_{i,k} \frac{\partial u^-}{\partial n}(t),
\]

\[
\lambda_{i,k} \frac{\partial u^-}{\partial n}(t) + \gamma_k (u^-(t) - u^+(t)) = 0, \quad t \in \bigcup_{m_1,m_2} T_k,
\]
where \( \gamma_k > 0 \) are the so-called resistance coefficients, the vector \( n = (n_1, n_2) \) is the outward unit normal vector to \( T_k \), and

\[
\frac{\partial}{\partial n} = n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y}
\]

is the outward normal derivative,

\[
u^+(t) := \lim_{z \to t, z \in D_0} u(z), \quad u^-(t) := \lim_{z \to t, z \in D_k} u(z), \; t \in T_k, \; k = 1, \ldots, N.
\]

We suppose additionally that the boundary functions \( u^+(t) \), \( u^-(t) \) are continuously differentiable on the corresponding curves. The conditions (7)–(8) form the so-called non-ideal contact conditions.

In addition, we assume that the external field is applied in the \( x \)-direction, and \( u \) has a constant jump in the \( x \)-direction:

\[
u(z + 1) = u(z) + 1, \quad u(z + i) = u(z).
\]

We introduce complex potentials \( \phi(z) \) and \( \phi_k(z) \) which are analytic in \( D_0 \) and \( D_k \), and continuously differentiable in the closures of \( D_0 \) and \( D_k \), respectively, by using the following relations

\[
u(z) = \begin{cases} 
\Re(\phi(z) + z), & z \in D_0, \\
\frac{2\lambda_m}{\lambda_m + \lambda_{i,k}} \Re \phi_k(z), & z \in D_k.
\end{cases}
\]

Thus, in particular, the function \( \phi(z) \) is double periodic in the following sense:

\[
\phi(z + 1) = \phi(z), \quad \phi(z + i) = \phi(z).
\]

The flux \( \nabla u(x, y) \) can be represented via derivatives of the complex potentials:

\[
\psi(z) := \frac{\partial \phi}{\partial z} = \frac{\partial u^+}{\partial x} - i \frac{\partial u^+}{\partial y} - 1, \quad z \in D_0,
\]

\[
\psi_k(z) := \frac{\partial \phi_k}{\partial z} = \frac{\lambda_m + \lambda_{i,k}}{2\lambda_m} \left( \frac{\partial u^-}{\partial x} - i \frac{\partial u^-}{\partial y} \right), \quad z \in D_k.
\]

Let us rewrite conditions (7)–(8) in terms of the complex potentials \( \phi(z) \) and \( \phi_k(z) \).

The boundary value of the normal derivative can be written in the form

\[
\frac{\partial u^-(t)}{\partial n} = n \cdot \nabla u^-(t) = \frac{2\lambda_m}{r_k(\lambda_m + \lambda_{i,k})} \Re[(t - a_k)(\phi_k)'(t)], \quad |t - a_k| = r_k.
\]

Let \( s \) be the natural parameter of the curve \( T_k \) and

\[
\frac{\partial}{\partial s} = -n_2 \frac{\partial}{\partial x} + n_1 \frac{\partial}{\partial y}
\]

be the tangent derivative along \( T_k \). Applying the Cauchy-Riemann equations

\[
\frac{\partial u^\pm}{\partial x} = \frac{\partial v^\pm}{\partial y}, \quad \frac{\partial u^\pm}{\partial y} = -\frac{\partial v^\pm}{\partial x},
\]
where \( v \) is the imaginary part of the function \( \varphi(z) \), we obtain

\[
\frac{\partial u^\pm}{\partial n} = \frac{\partial v^\pm}{\partial s}.
\] (18)

The equality (7) can be written as

\[
\lambda_m \frac{\partial v^+}{\partial s}(t) = \lambda_{i,k} \frac{\partial v^-}{\partial s}(t), \quad |t - a_k| = r_k.
\] (19)

Integrating the last equality on \( s \), we arrive at the relation

\[
\lambda_m v^+(t) = \lambda_{i,k} v^-(t) + c,
\] (20)

where \( c \) is an arbitrary constant. We put \( c = 0 \) since the imaginary part of the function \( \varphi \) is determined up to an additive constant which does not have impact on the form of \( u \). Using (12), we have

\[
\Re \varphi(t) = -3t + \frac{2\lambda_{i,k}}{\lambda_m + \lambda_{i,k}} \Re \varphi_k(t), \quad |t - a_k| = r_k.
\] (21)

Using (15), we are able to write the equality (8) in the following form:

\[
\Re \varphi(t) = \frac{2\lambda_{i,k}\lambda_m}{r_k \gamma_k(\lambda_m + \lambda_{i,k})} \Re[(t - a_k)(\varphi_k)'(t)] - \Re t + \frac{2\lambda_m}{\lambda_m + \lambda_{i,k}} \Re \varphi_k.
\] (22)

By adding the relation (22) and (21) multiplied by \( i \), and by using \( \Re \varphi_k = \frac{\varphi_k + \overline{\varphi_k}}{2} \), \( \Im \varphi_k = \frac{\varphi_k - \overline{\varphi_k}}{2i} \), \( t - a_k = \frac{r_k^2}{t - a_k} \), we obtain

\[
\varphi(t) = \varphi_k(t) - \rho_k \varphi_k(t) + \mu_k(t - a_k)(\varphi_k)'(t) + \mu_k \frac{r_k^2}{t - a_k} (\varphi_k)'(t) - t, \quad |t - a_k| = r_k,
\] (23)

where \( \rho_k = \frac{\lambda_{i,k} - \lambda_m}{\lambda_{i,k} + \lambda_m} \) and \( \mu_k = \frac{\lambda_m(\rho_k + 1)}{2r_k \gamma_k} \).

Let us now differentiate (23). First, we note that

\[
(\varphi(t))' = -\left( \frac{r_k}{t - a_k} \right)^2 \varphi'(t), \quad (\varphi'(t))' = -\left( \frac{r_k}{t - a_k} \right)^2 \varphi'(t), \quad |t - a_k| = r_k.
\] (24)

This can be easily shown by representing the function \( \varphi \) in the form \( \varphi(z) = \sum_{l=0}^{\infty} a_k (z - a_k)^l \), \( |z - a_k| \leq r_k \), and by using the relation \( t = \frac{r_k^2}{t - a_k} + a_k \) on the boundary \( |t - a_k| = r_k \). Thus, after differentiating (23), we arrive at the following

**R-linear boundary value problem** ([12]) on each contour \( |t - a_k| = r_k \),

\[
\psi(t) = (1 + \mu_k) \psi_k(t) + (\rho_k - \mu_k) \left( \frac{r_k}{t - a_k} \right)^2 \psi_k(t) + \mu_k (t - a_k) \psi_k'(t) - \mu_k \frac{r_k^2}{t - a_k} \psi_k'(t) - 1,
\] (25)

with \( k = 1, 2, \ldots, N \).

The formula for the effective conductivity tensor \( \Lambda_e \) of a composite material
represented by the cell \(Q_{(0,0)}\) has the following form

\[
\Lambda_e = \begin{pmatrix}
\lambda_x^e & \lambda_{xy}^e \\
\lambda_{x,y}^e & \lambda_y^e
\end{pmatrix}.
\]

The components of the tensor \(\Lambda_e\) of a composite material with \(N\) disjoint inclusions are defined by the formulas (see [2])

\[
\lambda_x^e = \lambda_m \int_{D_0} \frac{\partial u}{\partial x} \, dx dy + \sum_{k=1}^{N} \lambda_{i,k} \int_{D_k} \frac{\partial u}{\partial x} \, dx dy,
\]

\[
\lambda_{xy}^e = \lambda_m \int_{D_0} \frac{\partial u}{\partial y} \, dx dy + \sum_{k=1}^{N} \lambda_{i,k} \int_{D_k} \frac{\partial u}{\partial y} \, dx dy,
\]

where \(u(x, y)\) is the solution to the problem (6)–(8).

Let us denote by \(\partial D_k^+ (\partial D_k^-)\) the boundary of the \(k\)-th disc, oriented in the counter-clockwise (clockwise) direction, respectively. By Green’s formula we have

\[
\lambda_x^e = \lambda_m \int_{\partial Q_{(0,0)}} u^+ \, dy + \sum_{k=1}^{N} \int_{\partial D_k^-} (\lambda_m u^+ - \lambda_{i,k} u^-) \, dy
\]

\[
= \lambda_m \int_{\partial Q_{(0,0)}} u^+ \, dy + \lambda_m \sum_{k=1}^{N} \int_{\partial D_k^-} (u^+ - u^-) \, dy + \sum_{k=1}^{N} (\lambda_{i,k} - \lambda_m) \int_{\partial D_k^+} u^- \, dy (28)
\]

Analogously,

\[
\lambda_{xy}^e = \lambda_m \int_{\partial Q_{(0,0)}} u^+ \, dx + \sum_{k=1}^{N} \int_{\partial D_k^-} (\lambda_m u^+ - \lambda_{i,k} u^-) \, dx
\]

\[
= \lambda_m \int_{\partial Q_{(0,0)}} u^+ \, dx + \lambda_m \sum_{k=1}^{N} \int_{\partial D_k^-} (u^+ - u^-) \, dx + \sum_{k=1}^{N} (\lambda_{i,k} - \lambda_m) \int_{\partial D_k^+} u^- \, dx (29)
\]

By the relations (7) and (8) we have on the boundary of the \(k\)-th disc

\[
\frac{u^+ - u^-}{\gamma_k \, \frac{\partial u^-}{\partial n}} = \frac{\lambda_{i,k} \, \frac{\partial u^-}{\partial n}}{\gamma_k} = \frac{\lambda_m \, \frac{\partial u^+}{\partial n}}{\gamma_k}.
\]

At last, applying to each integral in the last sums of (28) and (29) Green’s formula
we obtain the following formula for effective conductivity in the considered case

\[
\lambda^x - i\lambda^xy = \lambda_m \int_{Q(0,0)} \left( \frac{\partial u^+}{\partial x} - i \frac{\partial u^+}{\partial y} \right) dxdy + \lambda_m \sum_{k=1}^{N} \frac{\lambda_m}{\gamma_k} \int_{D_k} \left( \frac{\partial u^+}{\partial n} - i \frac{\partial u^+}{\partial n} \right) dxdy + \sum_{k=1}^{N} (\lambda_{i,k} - \lambda_m) \int_{D_k} \left( \frac{\partial u^-}{\partial x} - i \frac{\partial u^-}{\partial y} \right) dxdy.
\]  

(30)

By using notation (14) and mean value theorem for analytic functions we can rewrite the last formula in terms of functions \( \psi \) and \( \psi_k \)

\[
\lambda^x - i\lambda^xy = \lambda_m \int_{Q(0,0)} \psi(z) dxdy + (-i) \sum_{k=1}^{N} \frac{\lambda_m^2}{\gamma_k} \int_{D_k} \frac{\partial u^+}{\partial n} dz + 2\lambda_m \sum_{k=1}^{N} \rho_k \nu_k \psi(a_k),
\]

(31)

where \( \rho_k = \frac{\lambda_{i,k} - \lambda_m}{(\lambda_{i,k} + \lambda_m)} \), \( \nu_k = \pi r_k^2 \).

**Remark 1**: When the external field is equal \( \tau \) then the first integral in (31) is equal to the area of the unit cell. Thus, in the case of the ideal contact conditions formula (31) coincides with Mityushev’s formula for effective conductivity of doubly periodic composite materials with an array of circular cylindrical inclusions (see, e.g. [12]).

3. Functional equations in spaces of analytic functions

In order to determine the effective conductivity we have to find first the above introduced functions \( \psi_k \), \( k = 1, 2, \ldots, N \). It means that we should solve the boundary value problem (25). For this we use the so-called *method of functional equations* (cf. [12]).

Notice that we have \( N \) contours \( T_k \) and \( N \) complex conjugation conditions on each contour \( T_k \) but we need to find \( N + 1 \) functions \( \psi, \psi_1, \ldots, \psi_N \). This means that we need one additional condition to close up the system. For this reason we introduce a new function \( \Phi \), which is analytic in \( Q(0,0) \) and in \( \bigcup_{k=1}^{N} D_k \), has the zero jumps along each \( T_k, \) \( k = 1, 2, \ldots, N \), and is doubly periodic on \( \mathbb{C} \). Then, by using Liouville’s theorem for doubly periodic functions, we have that \( \Phi(z) = c \) (for some constant \( c \)). Such consideration gives an additional condition on \( \psi, \psi_1, \ldots, \psi_N \).

We will now show that the just mentioned constant \( c \) turns out to be equal to zero. Let us introduce the function \( \Phi(z) \) by the following formulas:

\[
\Phi(z) = \begin{cases} 
\Phi(k)(z), & |z - a_k| \leq r_k, \\
\Phi(0)(z), & z \in D_0,
\end{cases}
\]

(32)
\[ \Phi_{(k)}(z) = (1 + \mu_k)\psi_k(z) + \mu_k(z - a_k)\psi'_k(z) - (\rho_k - \mu_k) \sum_{m=1}^{N} \sum_{m_1, m_2} \ast W_{m_1, m_2, m} \psi_m(z) \]
\[ + \mu_k \sum_{m=1}^{N} \sum_{m_1, m_2} \ast W_{m_1, m_2, m} \psi'_m(z) - 1, \quad (33) \]

\[ \Phi_{(0)}(z) = \psi(z) - (\rho_k - \mu_k) \sum_{m=1}^{N} \sum_{m_1, m_2} W_{m_1, m_2, m} \psi_m(z) + \mu_k \sum_{m=1}^{N} \sum_{m_1, m_2} W'_{m_1, m_2, m} \psi'_m(z). \quad (34) \]

Here,

\[ W_{m_1, m_2, k} \psi_k(z) = \left( \frac{r_k}{z - a_k - m_1 - im_2} \right)^2 \psi_k \left( \frac{r_k^2}{z - a_k - m_1 - im_2} + a_k \right), \quad (35) \]

\[ W'_{m_1, m_2, k} \psi'_k(z) = \frac{r_k^4}{(z - a_k - m_1 - im_2)^2} \psi'_k \left( \frac{r_k^2}{z - a_k - m_1 - im_2} + a_k \right), \quad (36) \]

\[ \sum_{m=1}^{N} \sum_{m_1, m_2} \ast W_{m_1, m_2, m} := \sum_{m \neq k} \sum_{m_1, m_2} W_{m_1, m_2, m} + \sum_{m_1, m_2} W'_{m_1, m_2, k}. \quad (37) \]

The “prime” notation in \( \sum \ast ' \) means that the summation occurs in all \( m_1 \) and \( m_2 \) except at \((m_1, m_2) = (0, 0)\).

For each fixed \( k = 1, 2, \ldots, N \), we define the Banach space \( C_k \) of the continuous functions on \( T_k \) with the usual norm \( \| \psi_k \| := \max_{T_k} |\psi_k(t)|, \) \( k = 1, 2, \ldots, N \), and the closed subspace \( C^1_+ \subset C_k \) of those functions \( \psi_k(z) \) which admit an analytic continuation into \( D_k \). Let \( C^1_+ \) be the subspace of \( C_k \) consisting of continuously differentiable functions on \( T_k \) endowed with the norm

\[ \| \psi_k \| := \max_{T_k} |\psi_k(t)| + \max_{T_k} |\psi'_k(t)|, \quad k = 1, 2, \ldots, N, \quad (38) \]

and consider also the closed subspace \( C^{1+}_k \subset C^1_+ \) of functions which admit an analytic continuation into \( D_k \). For each fixed \( k = 1, \ldots, N \), we also use the generalized Hardy space \( \mathcal{H}^2(D_k) \) of analytic functions on \( D_k \) satisfying the condition

\[ \sup_{0 < r < r_k} \int_{0}^{2\pi} |\psi_k(re^{i\theta} + a_k)|^2 d\theta < \infty. \quad (39) \]
\[ \|\psi_k\|_{\mathcal{H}^2(D_k)}^2 := \sup_{0 < r < r_k} \int_0^{2\pi} |\psi_k(re^{i\theta} + a_k)|^2 \, d\theta. \quad (40) \]

Note that \( C_{k}^{1+} \subset C_{k}^{+} \subset \mathcal{H}^2(D_k) \).

We introduce a new function \( \phi : \bigcup_{k=1}^{N} D_k \to \mathbb{C} \), \( \phi|_{D_k} = \psi_k \), which is piece-wise analytic in the disconnected domain \( \bigcup_{k=1}^{N} D_k \). We consider the Banach space \( \mathcal{H}^2 \) of functions \( \phi \) with “components” \( \psi_k \in \mathcal{H}^2(D_k) \) endowed with the norm \( \|\phi\|_{\mathcal{H}^2}^2 := \|\psi_1\|_{\mathcal{H}^2(D_1)}^2 + \cdots + \|\psi_N\|_{\mathcal{H}^2(D_N)}^2 \).

Let \( \alpha(z) := \frac{r_k^2}{z - a_k - m_1 - im_2} + a_k \). \quad (41) \]

If \( m_1 = m_2 = 0 \), then \( \alpha(z) \) becomes the symmetry with respect to the circle \( T_k \).
If \( m_1 + im_2 \neq 0 \), then \( \alpha(z) \) consists of the symmetry with respect to the circle \( T_k + m_1 + im_2 \) and translation by the vector \(-m_1 + im_2\). Hence, \( \alpha(z) \) with \( m_1 + im_2 \neq 0 \) transforms the closed disk \(|z - a_k| \leq r_k\) into another closed disk which is compactly embedded into \(|z - a_k| < r_k\). This means that \( \alpha(z) \) possesses a contraction property. This property allows us to assert that the operator \( W_{m_1, m_2, k} \) is compact from \( C_k^{+} \) into \( C_k^{+} \), and that \( W_{m_1, m_2, k}' \) is compact from \( C_k^{1+} \) into \( C_k^{+} \). Indeed, using the Cauchy integral formula, we have the following representations

\[ \psi_k \left( \frac{r_k^2}{z - a_k - m_1 - im_2} + a_k \right) = \frac{1}{2\pi i} \int_{T_k} \frac{\psi_k(\tau)\,d\tau}{\tau - \frac{r_k^2}{z - a_k - m_1 - im_2} - a_k}, \quad (42) \]

\[ \psi_k' \left( \frac{r_k^2}{z - a_k - m_1 - im_2} + a_k \right) = \frac{1}{2\pi i} \int_{T_k} \frac{\psi_k'(\tau)\,d\tau}{\tau - \frac{r_k^2}{z - a_k - m_1 - im_2} - a_k} \quad (43) \]

with continuous kernels as functions in \((\tau, z) \in T_k \times D_k\) (due to the contraction property), and thus gives the compactness of operators defined in (42) and (43). Therefore, (35) and (36) are compositions of the compact operators arising in (42) and (43), the operator of complex conjugation, and bounded operators of multiplication by \((r_k/(z - a_k - m_1 - im_2))^2\) and \(r_k^4/(z - a_k - m_1 - im_2)^3\), respectively. It follows that the operators in (35) and (36) are compact.

4. Analyticity of the functions and compactness of the operators

We show now that the function \( \Phi(z) \) possesses all the above mentioned properties. In particular, we justify here the analyticity of the involved functions, and the compactness of the corresponding constructed operators.
Let us expand $\psi_k(z)$ and $\psi'_k(z)$ into Taylor series

$$
\psi_k(z) = \sum_{l=0}^{\infty} \psi_k(z - a_k)^l, \quad \psi'_k(z) = \sum_{l=1}^{\infty} \psi_{lk} l (z - a_k)^{l-1}
$$

in order to sum up $W_{m_1, m_2, k} \psi_k(z)$ and $W'_{m_1, m_2, k} \psi'_k(z)$ over all translations $m_1 + m_2$ and obtain double periodic functions.

**Proposition 4.1:**

(i) The series $\sum_j W_{j,k} \psi_k(z)$ and $\sum_j W'_{j,k} \psi'_k(z)$, where $j = (m_1, m_2)$ and $k$ is a fixed number, converge absolutely and uniformly in the perforated cell $D_0 \cup \partial D_0$. They define functions which are analytic in $D_0$, continuous in $D_0 \cup \partial D_0$ and double periodic. These functions can be written in the form

$$
\sum_j W_{j,k} \psi_k(z) = \sum_{l=0}^{\infty} \psi_{lk} r_k^{2(l+1)} E_{l+2}(z - a_k), \quad (45)
$$

and

$$
\sum_j W'_{j,k} \psi'_k(z) = \sum_{l=1}^{\infty} \psi_{lk} l r_k^{2(l+1)} E_{l+2}(z - a_k), \quad z \in D_0, \quad (46)
$$

where $E_p(z)$ is the elliptic Eisenstein function of order $p$ (see [15]), defined by the formula $E_p(z) := \sum_{m_1, m_2} (z - m_1 - m_2)^{-p}$.

(ii) The series

$$
\sum_j W'_{j,k} \psi_k(z) := \sum_j W_{j,k} \psi_k(z) - \left( \frac{r_k}{z - a_k} \right)^2 \psi_k \left( \frac{r_k^2}{z - a_k} + a_k \right) \quad (47)
$$

and

$$
\sum_j W'_{j,k} \psi'_k(z) := \sum_j W'_{j,k} \psi'_k(z) - \frac{r_k^4}{(z - a_k)^3} \psi'_k \left( \frac{r_k^2}{z - a_k} + a_k \right) \quad (48)
$$

define analytic functions in the unit cell $Q_{(0,0)}$, and continuous in $Q_{(0,0)} \cup \partial Q_{(0,0)}$. These functions can be written in the form

$$
\sum_j W'_{j,k} \psi_k(z) = \sum_{l=0}^{\infty} \psi_{lk} r_k^{2(l+1)} \sigma_{l+2}(z - a_k), \quad (49)
$$

and

$$
\sum_j W'_{j,k} \psi'_k(z) = \sum_{l=1}^{\infty} \psi_{lk} l r_k^{2(l+1)} \sigma_{l+2}(z - a_k), \quad (50)
$$

respectively, where $\sigma_l$ is the modified Eisenstein function defined by the formula $\sigma_l(z) := E_l(z) - z^{-l}$, $l = 1, 2, \ldots$. 

(iii) The linear operator $\sum_j W_{j,k}'$ is compact in $C_k^1$, and the linear operator $\sum_j W_{j,k}$ is compact in $C_k^1$.

**Proof:** Let us prove proposition (i) for the series $\sum_j W_{j,k} \psi_k(z)$. This statement in the case of the series $\sum_j W_{j,k} \psi_k(z)$ can be proved analogously. Therefore, the corresponding proof is omitted here.

Let $\{e_j\}_{j=0}^\infty$ be a linear ordered set formed by the numbers $m_1 + m_2$ and with $e_0 = 0$. In addition, consider the series $\sum_{j=M}^\infty W_{j,k}' \psi_k(z)$, where the number $M$ is chosen as follows: $|z - a_k - e_j| \geq h > 1$ for all $j \geq M$ and $z - a_k \in Q_{(0,0)}$. Using the Taylor expansion (44) of the function $\psi_k(z)$, we have

$$\sum_{j=M}^\infty W_{j,k}' \psi_k(z) = \sum_{j=M}^\infty \sum_{l=1}^\infty \psi_{lk} l r_k^{2(l+1)} (z - a_k - e_j)^{-3} (z - a_k - e_j)^{-l+1}. \quad (51)$$

The series $V(z) = \sum_{j=M}^\infty |z - a_k - e_j|^{-3}$ converges uniformly in $D_0 \cup \partial D_0$ (cf. [7]).

Since $|z - a_k - e_j|^{-l+1} \leq h^{-l+1}, l = 1, 2, \ldots$, and $r_k < 1$, we have

$$\sum_{j=M}^\infty \sum_{l=1}^\infty \left| \psi_{lk} l r_k^{2(l+1)} (z - a_k - e_j)^{-3} (z - a_k - e_j)^{-l+1} \right| \leq h G \sum_{l=1}^\infty |\psi_{lk}| l h^{-l}, \quad (52)$$

where $G := \max_{D_0 \cup \partial D_0} |V(z)|$. Since $\psi_k \in C_k^1$ and $h > 1$, the series $\sum_{l=1}^\infty |\psi_{lk}| l h^{-l}$ is convergent. Therefore, the series (51) and hence the series (46) converge absolutely and uniformly in $D_0 \cup \partial D_0$.

The analyticity of the functions (45) and (46) in $D_0$, their continuity in $D_0 \cup \partial D_0$ and the doubly periodicity follow from the corresponding properties of the Eisenstein functions $E_p(z)$.

To prove proposition (ii) we exclude from the series $\sum_j W_{j,k} \psi_k(z)$ and $\sum_j W_{j,k}' \psi_k(z)$ an item containing the pole. Then it follows that the functions $\sum_j W_{j,k} \psi_k(z)$ and $\sum_j W_{j,k}' \psi_k(z)$ are analytic in the unit cell $Q_{(0,0)}$ and continuous in $Q_{(0,0)} \cup \partial Q_{(0,0)}$. The relations (49) and (50) follow from (44) and the definition of $\sigma_l(z)$. Thus, proposition (ii) is proved.

Since a converging series of compact operators converges to a compact operator, we conclude that the linear operator $\sum_j W_{j,k}$ is compact in $C_k^1$. Analogously, the linear operator $\sum_j W_{j,k}'$ is a compact operator in $C_k^1$. \qed
5. A consequence of the generalized Liouville theorem

According to (25) the jump of $\Phi(z)$ across $T_k$ takes the value

$$\Delta_k := \Phi(0)(t) - \Phi(k)(t) = \psi(t) - (\rho_k - \mu_k) \left( \frac{r_k}{t - a_k} \right)^2 \psi_k(t) + \mu_k \frac{r_k^4}{(t - a_k)^3} \psi_k'(t)$$

$$- (1 + \mu_k)\psi_k(t) - \mu_k(t - a_k)\psi_k'(z) + 1 = 0, \quad |t - a_k| = r_k, \quad (53)$$

Using the generalized Liouville theorem for double periodic functions, we conclude that $\Phi(z) \equiv c$, where $c$ is a constant. Let $\psi$ and $\psi_k$ be solutions of the system $\Phi(z) = c$. Then, we have

$$\psi(z) = \tilde{\psi}(z) + c, \quad \in D_0, \quad (54)$$

with some double periodic function $\tilde{\psi}(z)$. Inserting the last equality in (14) and then in (12), we obtain

$$u(z) = \Re(\tilde{\varphi}(z) + cz + z), \quad z \in D_0$$

with some double periodic function $\tilde{\varphi}(z)$ which together with (11) yields $c = 0$. Thus, we have $\Phi(z) \equiv 0$. Writing $\Phi(z) \equiv 0$ in $|z - a_k| \leq r_k$, we obtain the following system of linear functional equations

$$\psi_k(z) = -\frac{\mu_k}{1 + \mu_k} (z - a_k)\psi_k'(z) + \frac{\rho_k - \mu_k}{1 + \mu_k} \sum_{m=1}^{N} \sum_{j} *W_{j,m} \psi_m(z)$$

$$- \frac{\mu_k}{1 + \mu_k} \sum_{m=1}^{N} \sum_{j} *W'_{j,m} \psi_m'(z) + \frac{1}{1 + \mu_k}, \quad k = 1, 2, \ldots, N, \quad (56)$$

with respect to $\psi_k(z) \in C_k^{1+}$. The corresponding function $\psi(z)$ has the form

$$\psi(z) = \sum_{k=1}^{N} (\rho_k - \mu_k) \sum_{m=1}^{N} \sum_{j} W_{j,m} \psi_m(z) - \sum_{k=1}^{N} \sum_{m=1}^{N} \sum_{j} W'_{j,m} \psi_m'(z). \quad (57)$$

The system (56) can be considered as an equation for the function $\Psi(z)$ in the space $H_2^2$:

$$\Psi(z) = -\tilde{\mu} \frac{z - \tilde{a}}{1 + \tilde{\mu}} \Psi'(z) + \tilde{\rho} - \tilde{\mu} \sum_{m=1}^{N} \sum_{j} *W_{j,m} \Psi(z)$$

$$- \frac{\tilde{\mu}}{1 + \tilde{\mu}} \sum_{m=1}^{N} \sum_{j} *W'_{j,m} \Psi'(z) + \frac{1}{1 + \tilde{\mu}}, \quad z \in \bigcup_{k=1}^{N} (D_k \cup T_k), \quad (58)$$

where $\Psi(z) = \psi_k(z)$ in $|z - a_k| \leq r_k$ for each $k = 1, \ldots, N$, and $\tilde{\mu} = \mu_k, \tilde{\rho} = \rho_k, \tilde{a} = a_k$, for $k = 1, \ldots, N$. 
6. On the solvability of the problem and characterization of the solution

Let us introduce on $\mathcal{C}^{1+}$ the operator $E$ defined by the following formulas

$$Ef(z) = f(z) + \frac{\tilde{\mu}}{1 + \tilde{\mu}} (z - \tilde{a}) f'(z), \quad |z - a_k| \leq r_k, \quad k = 1, \ldots, N, \quad (59)$$

where $\tilde{\mu} = \mu_k$, $\tilde{a} = a_k$ for $z \in D_k$, $k = 1, \ldots, N$.

Let us also introduce the operator $A_k$ in $\mathcal{C}^{1+}$ by the following formula

$$A_k \psi_k(z) = \left( \rho_k - \mu_k \right) \sum_{m=1}^{N} \sum_{j} W_{j,m} \psi_m(z) - \mu_k \sum_{m=1}^{N} \sum_{j} W'_{j,m} \psi'_m(z), \quad |z - a_k| \leq r_k,$$

and the operator $A = A_k, k = 1, \ldots, N$. The operator

$$T := I - \frac{1}{1 + \tilde{\mu}} E^{-1} A \quad (60)$$

characterizes the equation (58). Indeed, the system of equations (56) can be rewritten in the form of the following operator equation in $\mathcal{H}^2$:

$$E \Psi = \frac{1}{1 + \tilde{\mu}} \Psi + g, \quad (61)$$

where the function

$$g(z) \equiv \frac{1}{1 + \mu_k}, \quad z \in D_k, \quad (62)$$

is a constant in each domain $D_k$. Equation (61) is now equivalently rewritten as

$$\Psi = \frac{1}{1 + \tilde{\mu}} E^{-1} A \Psi + h \quad (63)$$

and as

$$T \Psi = h \quad (64)$$

with $h = E^{-1} g$.

Thus, it is possible to characterize the solvability of equation (58) (or equation (64)) by studying the kernel and cokernel of this operator $T$.

**Proposition 6.1:** If $\frac{\mu_k}{1 + \mu_k} > 0$, $k = 1, \ldots, N$, then the operator $T$ defined in (60) has a kernel and cokernel of finite dimension and these dimensions coincide.

**Proof:** It is shown in [4] that for $\frac{\mu_k}{1 + \mu_k} > 0$ the operator $E^{-1}$ is bounded in $\mathcal{H}^2$ and $\|E^{-1}\| \leq 1$. Thus, the corresponding conclusions are valid in our case too. Besides, if $f \in C^{1+}_k$ then $E^{-1} f \in C^{1+}_k$ (see also [4]).

By Proposition 4.1, (iii), the operator $A_k$ is compact in $C^{1+}_k$. Therefore, the operator $E^{-1} A$ is also a compact operator in $\mathcal{H}^2$ for $\frac{\mu_k}{1 + \mu_k} > 0$ (as a multiplication of compact and bounded operators).

As a consequence, the structure of the operator $T$ (being the identity plus a compact operator) ensures that $T$ is a Fredholm operator with zero Fredholm
Theorem 6.2: For sufficiently small parameters $|\rho_k|$ and $\mu_k > 0$, $k = 1, \ldots, N$, the equation (58) has a unique solution in $\mathcal{H}^2$.

Proof: From Proposition 6.1, we know already that in the conditions of the theorem the kernel and cokernel of the operator $T$ have equal finite dimension. In addition, we take into account that $E^{-1}$ is a contractive operator and that $\mu_k > 0$ also implies $\left| \frac{1}{1+\mu_k} \right| < 1$. Let us choose $\mu_k > 0$ and $|\rho_k|$ small enough such that $\|A_k\| < 1$. Then from (63) we conclude by the method of successive approximations that the functional equation (58) has a unique solution in $\mathcal{H}^2$.

Remark 1: Let us note that we actually construct $\psi_k$ elements which are continuously differentiable in $|z - a_k| \leq r_k$. This occurs due to the fact that the function $g$ defined by (62) is continuously differentiable in $|z - a_k| \leq r_k$. In addition, $E^{-1}g \in C^1_k$, and the operator $E^{-1}A$ transforms $\mathcal{H}^2$ into $C^1_k$.

Note also that the convergence in $C^+_k$ means uniform convergence which preserves analyticity in the limit. The operators $W_{m_1,m_2,m}$ and $W'_{m_1,m_2,m}$ depend analytically on $r_k^2$. Thus, we can look for $\psi_k(z)$ in the form of a series expansion in $r_k^2$:

$$\psi_k(z) = \sum_{s=0}^{\infty} \psi_k^{(s)}(z) r_k^{2s}. \quad (65)$$

In addition,

$$\psi'_k(z) = \sum_{s=0}^{\infty} [\psi_k^{(s)}(z)]' r_k^{2s}. \quad (66)$$

Substituting (65) into (31), we can write the effective conductivity $\lambda_e$ in the form of a series in the concentration of inclusions $\nu_k$:

$$\lambda_e = \lambda_m + 2\lambda_m \sum_{k=1}^{N} \rho_k \nu_k [A_0^{(k)} + A_1^{(k)} \nu_k + A_2^{(k)} \nu_k^2 + \ldots], \quad (67)$$

where the coefficients $A_s^{(k)}$ are defined by the equalities

$$A_s^{(k)} = \frac{1}{\pi^s} \psi_k^{(s)}(a_k), \quad s = 0, 1, 2, \ldots. \quad (68)$$

In particular, in the case of circular inclusions of equal radii ($r_k = r$, $k = 1, \ldots, N$) filled in by material of equal conductivity ($\lambda_{i,k} = \lambda_i$; $k = 1, \ldots, N$) this formula has the form

$$\lambda_e = \lambda_m + 2\lambda_m \rho \nu [A_0 + A_1 \nu + A_2 \nu^2 + \ldots], \quad (69)$$

where $\nu = N \pi r^2$ is the common concentration of inclusions in the unit cell, and the coefficients $A_s$ are defined by the equalities

$$A_s = \frac{1}{\pi^s N^{s+1}} \sum_{k=1}^{N} \psi_k^{(s)}(a_k), \quad s = 0, 1, 2, \ldots. \quad (70)$$
Using the representations (45)–(50), we are now able to write (56) in the form
\[
\sum_{s=0}^{\infty} \psi_k^{(s)}(z) r_k^{2s} = -\frac{\mu_k}{1 + \mu_k} (z - a_k) \sum_{s=0}^{\infty} \left[ \psi_k^{(s)}(z) \right]' r_k^{2s}
\]
\[+ \frac{\rho_k - \mu_k}{1 + \mu_k} \sum_{m\neq k} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \psi_{lm}^{(s)} r_m^{2(l+s+1)} E_{l+2}(z - a_m)
\]
\[+ \frac{\rho_k - \mu_k}{1 + \mu_k} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \psi_{lk}^{(s)} r_k^{2(l+s+1)} \sigma_{l+2}(z - a_k)
\]
\[- \frac{\mu_k}{1 + \mu_k} \sum_{m\neq k} \sum_{l=1}^{\infty} \sum_{s=0}^{\infty} \psi_{lm}^{(s)} l r_m^{2(l+s+1)} E_{l+2}(z - a_m)
\]
\[- \frac{\mu_k}{1 + \mu_k} \sum_{l=1}^{\infty} \sum_{s=0}^{\infty} \psi_{lk}^{(s)} l r_k^{2(l+s+1)} \sigma_{l+2}(z - a_k) + \frac{1}{1 + \mu_k},
\]
where each term in (65) in expanded into the Taylor series
\[
\psi_k^{(s)}(z) = \sum_{l=0}^{\infty} \psi_k^{(s)}_{lk}(z - a_k)^l.
\] (71)

Collecting the coefficients of the same power of \( r_k^2 \), we obtain the following theorem:

**Theorem 6.3:** Let \( u(z) \) be the solution of the problem (6)–(8) and let \( \nabla u \) be the flux defined in (14)
\[
\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \begin{cases} 
\psi(z) + 1, & z \in D_0 \cup \partial D_0, \\
\frac{2\lambda_m}{\lambda_m + \lambda_{ik}} \psi_k(z), & z \in D_k \cup T_k.
\end{cases}
\] (72)

Here, \( z = x + iy \) and \( \psi(z) \) and \( \psi_k(z) \) are given by (57) and (65), respectively, where \( \psi_k^{(p)}(z) \) are defined by the following recurrence relations:
\[
\psi_k^{(0)}(z) = -\frac{\mu_k}{1 + \mu_k} (z - a_k) [\psi_k^{(0)}(z)]' + \frac{1}{1 + \mu_k}
\]
\[
\psi_k^{(1)}(z) = -\frac{\mu_k}{1 + \mu_k} (z - a_k) [\psi_k^{(1)}(z)]' + \frac{\rho_k - \mu_k}{1 + \mu_k} \left[ \sum_{m \neq k} \psi_{0m}^{(0)} E_2(z - a_m) + \psi_{0k}^{(0)} \sigma_2(z - a_k) \right]
\]
\[
\psi_k^{(p)}(z) = -\frac{\mu_k}{1 + \mu_k}(z - a_k)[\psi_k^{(p)}(z)]' + \frac{\rho_k - \mu_k}{1 + \mu_k} \left[ \sum_{m \neq k} \psi_{0m}^{(p-1)} E_2(z - a_m) \right. \\
+ \left. \psi_{0k}^{(p-1)} \sigma_2(z - a_k) + \sum_{m \neq k} \psi_{1m}^{(p-2)} E_3(z - a_m) + \psi_{1k}^{(p-2)} \sigma_3(z - a_k) \right] + \cdots + \sum_{m \neq k} \psi_{p-1m}^{(0)} E_{p-1}(z - a_m) + \psi_{p-1k}^{(0)} \sigma_{p-1}(z - a_k) \right] + \sum_{m \neq k} 2 \psi_{m}^{(p-3)} E_4(z - a_m) + 2 \psi_{2m}^{(p-3)} \sigma_4(z - a_k) + \cdots + \sum_{m \neq k} (p-1) \psi_{p-1m}^{(0)} E_{p-1}(z - a_m) + (p-1) \psi_{p-1k}^{(0)} \sigma_{p-1}(z - a_k) \right] ,
\]

where \( p = 2, 3, \ldots \). Here, \( \psi_{lm}^{(s)} \) is the derivative of order \( l \) of \( s \)-th approximation.

Note that each equation in the system of differential equations (73) has the form:

\[
\psi_k^{(p)}(z) = -\frac{\mu_k}{1 + \mu_k}(z - a_k)[\psi_k^{(p)}(z)]' + q(z),
\]

where \( q \) is a doubly periodic function. Substituting \( \psi_k^{(p)}(z) = (z - a_k)^{-\frac{\mu_k}{\rho_k} + 1} u(z) \) into (74), we obtain

\[
\psi_k^{(p)}(z) = (z - a_k)^{-\frac{\mu_k}{\rho_k}} \left[ 1 + \frac{\mu_k}{\mu_k} \int (z - a_k)^{\frac{1}{\rho_k}} q(z) \, dz + c_p \right],
\]

where \( c_p \) is an arbitrary constant. Since we are looking for double periodic solutions, the term \( c(z - a_k)^{-\frac{\mu_k}{\rho_k}} \) should disappear. It yields \( c_p = 0 \).

Note also that if a value of \( \mu_k \) is a rational number, then the integral on the right-hand side of (75) can be represented through elementary functions. Finally, note that the two first coefficients \( \psi_k^{(0)}(a_k), \psi_k^{(1)}(a_k) \) in the formula (68) can be easily found. Namely, we have

\[
\psi_k^{(0)}(a_k) = \frac{1}{1 + \mu_k}, \quad \psi_k^{(1)}(a_k) = \frac{\rho_k - \mu_k}{(1 + \mu_k)^2} \left[ \sum_{m \neq k} E_2(a_k - a_m) + \sigma_2(0) \right].
\]

For a small concentration \( \nu_k \), it is enough to calculate several first terms \( \psi_k^{(p)}(a_k) \) for the coefficients \( A_p^{(k)} \) defined by (68). Formulas for the values of \( E_p(a_k - a_m) \) and \( \sigma_p(0) \) are presented, e.g., in [2].

Thus, we found a recursive algorithm to define the values \( \psi_k(a_k) \) and hence to determine the effective conductivity in terms of basic parameters of the considered model such as the conductivity of the matrix and inclusions, resistance coefficients,
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