PORTFOLIO DISCRETE OPTIMIZATION PROBLEMS

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HISTORICAL CHRONOLOGY OF PORTFOLIO OPTIMIZATION

1900 – Theory of Speculation, Bachelier L. (doctoral thesis)


1972 – Cost of Capital, Miller M. (Nobel Prize in Economics in 1990)


1976 – Start of Mathematical Phase, Elton E., Gruber M., Padberg M.

...
IMPORTANT MONOGRAPHS OF PORTFOLIO OPTIMIZATION

Google.com (10.05.2010):

“Portfolio Selection” – 2.980.000 links
“Modern Portfolio Optimization Theory” – 1.600.000 links
“Portfolio Management” – 23.300.000 links
The Markowitz portfolio model
(model based on Euclidean metric of risk)

j=1,…n is assets, C is capital investor constraint, (C=1)

\[ \sum_{j=1}^{n} x_j = C, \quad x_j \geq 0, \quad j = 1,...,n. \]

r_{j1}, … r_{jT} is historical returns, expected return \( r_j = E(R_j) = \sum_{t=1}^{T} p_t r_{jt} \)

\[ r(x) = E(r(x)) = \sum_{j=1}^{n} x_j r_j \Rightarrow \max, \]

OPTIMAL SOLUTION IS TRIVIAL: greedy=0…, C, …0

\[ \sigma(x) = \sqrt{E[R(x) - r(x)]^2} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j} \]

standard deviation

\[ \text{variance} \quad \sigma(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j \rightarrow \min \]

1. \( \lambda \) – Constraint Approach: \( \sigma(x) \rightarrow \min, \ r(x) \geq \lambda \)
2. Bicriterion Optimization Approach: moves to one-criterion parametric convex quadratic optimization

\[ \lambda r(x) - (1 - \lambda) \sigma(x) \rightarrow \max \]

Karush – Kuhn – Tucker conditions
3. Parametric Quadratic Programming Approach
\( \lambda_1 = 0, \ldots \lambda_k = 1 \)
Efficient Portfolio Frontier (Pareto front)

Merton 1972 Approach for Analytical Derivation Piecewise-Linear Trajectory

\[ X^\mu = \mu x' + (1 - \mu)x'' \]

\[ r(x^\mu) = \mu r(x') + (1 - \mu)r(x'') \]
## EXAMPLE

### Average annual return (%)

<table>
<thead>
<tr>
<th>Assets</th>
<th>$r_{jt}$</th>
<th>$r_j = E(R_j) = \sum_{i=1}^{6} r_{ji}/6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Blue chips</td>
<td>$x_1$</td>
<td>18.24</td>
</tr>
<tr>
<td>Hi-Tech shares</td>
<td>$x_2$</td>
<td>12.24</td>
</tr>
<tr>
<td>Real estate Market</td>
<td>$x_3$</td>
<td>8.23</td>
</tr>
<tr>
<td>Treasury Bonds</td>
<td>$x_4$</td>
<td>8.12</td>
</tr>
</tbody>
</table>

The covariance matrix $\sigma$ is given by:

$$\sigma = \begin{pmatrix}
29.0552 & 40.3909 & -0.2879 & -1.9532 \\
40.3909 & 267.344 & 6.8337 & -3.6970 \\
-0.2879 & 6.8337 & 0.3759 & -0.0566 \\
-1.9532 & -3.6970 & -0.0566 & 0.1597
\end{pmatrix}$$

The objective function $\sigma(x) = x\alpha x = 29.0552x_1^2 + 80.7818x_2x_1 - 0.5758x_3x_1 - 3.9064x_4x_1 + 267.344x_2^2 + 0.3759x_3^2 + 0.1597x_4^2 + 13.6673x_2x_3 - 7.3940x_2x_4 - 0.1133x_3x_4 \rightarrow \min$

subject to:

- $x_1 + \ldots + x_4 = 1$, $x_j \geq 0$, $j=1,\ldots,4$
- $10.6483x_1 + 11.98x_2 + 8.34x_3 + 8.6317x_4 \geq \lambda$
- $\lambda = 0, 0.2, \ldots, 1$
# PARETO OPTIMAL PORTFOLIOS
*(Parametric Quadratic Programming)*

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$x_1$ (%)</th>
<th>$x_2$ (%)</th>
<th>$x_3$ (%)</th>
<th>$x_4$ (%)</th>
<th>$r$</th>
<th>$\sqrt{\sigma}$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>11,98</td>
<td>16,35</td>
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<tr>
<td>0,8</td>
<td>91</td>
<td>9</td>
<td>0</td>
<td>0</td>
<td>10,77</td>
<td>5,7</td>
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<tr>
<td>0,5</td>
<td>7</td>
<td>1</td>
<td>0</td>
<td>92</td>
<td>8,78</td>
<td>0,15</td>
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<td>0,2</td>
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<td>0</td>
<td>16</td>
<td>79</td>
<td>8,69</td>
<td>0,09</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>0</td>
<td>18</td>
<td>77</td>
<td>8,68</td>
<td>0,088</td>
</tr>
</tbody>
</table>
Model with risk-free asset

Tobin (1958)

Risk-free asset $x_0$ hypothetically corresponds government securities:

$$x_0 + \sum_{j=1}^{n} x_j = C, \quad x_j \geq 0, \quad j = 0, \ldots, n$$

$$r(x_0, x) = r_0 x_0 + r(x) = r_0 x_0 + \sum_{j=1}^{n} r_j x_j \rightarrow \text{max}$$

$$\sigma = \sqrt{x_0^2 \sigma_0^2 + x_p^2 \sigma_p^2 + 2x_0 x_p \sigma_{0p}} = \sqrt{x_0^2 0 + \sigma_p^2 x_p^2 + 2x_0 x_p 0} = \sqrt{x_p^2 \sigma_p^2} = x_p \sigma_p = \sigma(x) \rightarrow \text{min}$$

Efficient Portfolio Frontier
Model of diversification of investor capital with submodular constraints

Investor can be expressed as simple constraints:
lower and upper bounds on individual assets \( j \): \( c_j \leq x_j \leq C_j \)
section constraints, i.e. constraints are invested in certain sectors \( I \) assets:
in shares of the energy sector at least 60% of the capital \( C \): \( \sum x_j \leq 0.6C \)
regional constraints invested in European shares at most 40% of the capital \( C \): \( \sum x_j \geq 0.4C \)
budget constraints and diversification constraints modeling sub (super) modular functions \( C(I) \), \( c(I) \):

\[
c(I) \leq \sum_{j \in I} x_j \leq C(I) \quad \text{for } I
\]

\( x_j \) is integer

Greedy solution is optimal solution in the auxiliary problem:

\[
\sum_{j=1}^{n} r_j x_j \rightarrow \max
\]

\( x \in \text{INVESTOR POLYMATROID} \)
Model with fractional utility criteria
Sharpe (1963, Manag. Sci.)

1) \[ \frac{\sum_{j=1}^{n} r_j x_j}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j}} \rightarrow \text{max} \]

2) \[ \frac{\sum_{j=1}^{n} r_j x_j}{\sum_{j=1}^{n} \beta_j x_j} \rightarrow \text{max}, \]

\( \beta_j \) is "beta-coefficient", i.e., covariance between return asset \( j \) and market index (f.ex. DAX)

\[ x \in \text{INVESTOR POLYMATROID} \]

Faaland and Jacob (1981, Manag. Sci.) developed Sharpe model with fractional-linear function utility criteria:

\[ \frac{\sum_{j=1}^{n} r_j x_j}{\sum_{j=1}^{n} \beta_j x_j} \quad \text{and supplement constraints: } x_j \leq \frac{1}{k}, j = 1 \ldots n \]

Algorithm: succession of greedy solutions:

\[ \frac{\sum_{j=1}^{k} r_j}{\sum_{j=1}^{k} \beta_j}, k = 1 \ldots n \]
Model based on Minkowski absolute metric $L_1$ of risk

$$\sigma(x) = E|R - E(R)| = E\left[\sum_{j=1}^{n} r_j x_j - E\left[\sum_{j=1}^{n} r_j x_j \right]\right] = \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{j=1}^{n} (r_{jt} - r_j) x_j \right|$$

Model Konno and Yamazaki (1991)

$$\sum_{t=1}^{T} y_t \rightarrow \text{min}$$

$$y_t + \sum_{j=1}^{n} (r_{jt} - r_j) x_j \geq 0, \quad t = 1, \ldots, T$$

$$y_t - \sum_{j=1}^{n} (r_{jt} - r_j) x_j \geq 0, \quad t = 1, \ldots, T.$$  

$x \in \text{INVESTOR POLYMATROID}$

Model Feinstein and Thapa (1993)

$$\text{min} \sum_{t=1}^{T} (u_t + v_t)$$

$$u_t + v_t - \sum(r_{jt} - r_j) x_j \geq 0, \quad t = 1, \ldots, T$$

$$u_t, v_t \geq 0$$

$x \in \text{INVESTOR POLYMATROID}$

where $p_u$ and $p_d$ denote penalty parameters.
Model based on Minkowski semi-absolute metric of risk

Model penalizes only the negative deviations:

\[ \sigma(x) = \frac{1}{T} \sum_{t=1}^{T} \min \left\{ 0, \sum_{j=1}^{n} (r_{jt} - r_j) x_j \right\} \]

Model Konno

\[ \min \sum_{t=1}^{T} y_t \]

\[ y_t + \sum (r_{jt} - r_j) x_j \geq 0, \quad y_t \geq 0, \quad t = 1..T \]

\[ x \in \text{INVESTOR POLYMATROID} \]
Model based on Chebyshev metric $l_\infty$ of risk

The minimax portfolio is defined as being optimal with respect to the historical dataset \( \{r_{jt}\} \)

The investor has a strong absolute aversion to down side risk

\[
\sigma(x) = \min_{t = 1 \ldots T} \sum_{j=1}^{n} r_{jt} x_j \rightarrow \max
\]

**Young Minimax model (1998)**

\[
\lambda \rightarrow \max \quad \sum_{j=1}^{n} r_{jt} x_j \geq \lambda, \quad t = 1, \ldots, T
\]

\[
\sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0
\]

\( x \in \text{INVESTOR POLYMATROID} \)

**Theorem (Konno, Yamazaki)**

\[
\sigma_{\infty}(x) = \max_{j=1 \ldots n} \sqrt{\frac{2}{\pi} \sigma_j x_j}
\]
Model with limited number assets
(cardinality constraints)

\[ \sum_{j=1}^{n} \delta_j \leq m, \quad \delta_j = 0 \quad \text{or} \quad 1, \quad x_j \leq \delta_j, \quad j = 1, \ldots, n, \]

Non polymatroidal constraints
Model with buy-in thresholds

\[ l_j \delta_j \leq x_j, \; \delta_j = 0 \text{ or } 1, \; j = 1, \ldots, n \]  

(thresholds constraints)

German Investment Law uses constraint (5, 10, 40):

“Securities of the same issuer are allowed to amount to up to 5% of the net asset value of the mutual fund. They are allowed to amount to 10%, however, if the total of all of these assets is less than 40% of the net asset value”

\[ \sum_{j=1}^{n} x_j \delta_j \leq 0.4, \]
\[ x_j - 0.05\delta_j \leq 0.05, \; \delta_j = 0 \text{ or } 1, \; j = 1, \ldots, n \]
Model with integer (lot) assets

asset $j$ has actual price $c_j$ or asset $j$ sells by lots in quantity $c_j$, $2c_j$, $3c_j$, …

$y_j$ indicates integer quantity of the asset $j$

$$c \leq \sum_{j=1}^{n} p_j y_j \leq C, \quad y_j \geq 0 \text{ and integer} \quad j = 1, \ldots, n$$

Non polymatroidal constraints

Mansini and Speranza (1999)

Models with transactions costs

\[
\sum_{j=1}^{n} (r_j - d_j)x_j \rightarrow \max \quad \text{(linear costs)}
\]

\[
\sum_{j=1}^{n} (r_jx_j - f_j \delta_j) \rightarrow \max \quad \text{(fixed cost)}
\]

\[x_j \leq \delta_j, \quad \delta_j = 0 \text{ or } 1 \quad j = 1, \ldots, n.\]
Model using fuzzy expected return


Investor examine $T$ potential market scenario

$$R_t (x) = \sum_{j=1}^{n} r_{jt} x_j , t = 1, \ldots, T$$

($r_{jt}$ – return from $j$ asset for the $t$ market scenario)

Fuzzy membership function $M_t (R_t (x))$ as modeling the fuzzy utility (degree of satisfaction) of the investor

$$\max \lambda$$

$$M_t (R_t (x)) \geq \lambda, t = 1, \ldots, T$$

$x=(x_1, \ldots, x_n) \in \text{INVESTOR POLYMATROID}$
Model with trapezoidal fuzzy return membership function

\[
\sum_{j=1}^{n} \left( a_{uj} - a_{lj} + \frac{1}{3} \left( c_j + d_j \right) \right) x_j \rightarrow \min \\
\sum_{j=1}^{n} \left( \frac{1}{2} (a_{uj} + a_{lj}) + \frac{1}{6} (d_j - c_j) \right) x_j \rightarrow \max
\]

if \( a_i = a_u \) is a triangular fuzzy member

if all return \( r_j \) is triangular, than conditions are found out, when greedy solution is optimal

Fang Y., Lai K., Wang S.

Fuzzy Portfolio Optimization


v. 609. 173 p.
A Solver for Portfolio Discrete Optimization is MIPCL

MIPCL (Mixed Integer Programming Class Library) is a tool for solving LP, MIP, QIP using MPS format, rules column generators, greedy heuristics for polymatroidal constraints, branching techniques for pure nonconvex constraints.

(developed in Belarusian State University by Pisaruk N.)

Application in collaboration with Belarusian IT-outsourcing companies and international banks, FINANCIAL REUTERS.
**HOW GOOD IS MIPCL?**

Comparison with SCIP 1.1.0

Test from [http://miplib.zib.de](http://miplib.zib.de)

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</table>
FUTURES RESEARCHES


- Multiple Criteria in Portfolio Optimization Problems (liquidity, social risk…) ex. IMF Recommendations for central banks: 4 criteria

- Multiple currency assets (USD, EUR, CNY, YEN …) and penalty FOREX (exchange loss)
THANK YOU FOR YOUR ATTENTION!