On $\tau$-closed totally saturated group formations with Boolean sublattices

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Abstract. In the universe of finite groups the description of $\tau$-closed totally saturated formations with Boolean sublattices of $\tau$-closed totally saturated subformations is obtained. Thus, we give a solution of Question 4.3.16 proposed by A. N. Skiba in his monograph "Algebra of Formations" (1997).

Introduction

All groups considered are finite. Used notations and terminology are standard (see [1]–[4]). Recall that a formation $\mathfrak{F}$ is called saturated if $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$. It is known [4] that if $\mathfrak{F}$ is a non-empty saturated formation, then $\mathfrak{F} = LF(f)$, i. e., $\mathfrak{F}$ has a local satellite $f$.

Every group formation is considered as 0-multiply saturated [5]. For $n \geq 1$, a formation $\mathfrak{F} \neq \emptyset$ is called $n$-multiply saturated [5], if it has a local satellite $f$ such that every non-empty value $f(p)$ of $f$ is a $(n-1)$-multiply saturated formation. A formation is called totally saturated [5] if it is $n$-multiply saturated for all natural $n$.

Let $\tau$ be a function such that for any group $G$, $\tau(G)$ is a set of subgroups of $G$, and $G \in \tau(G)$. Following [3] we say that $\tau$ is a subgroup functor if for every epimorphism $\varphi : A \rightarrow B$ and any groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^\varphi \in \tau(B)$ and $T^\varphi^{-1} \in \tau(A)$.

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A group class $\mathfrak{F}$ is called $\tau$-closed if $\tau(G) \subseteq \mathfrak{F}$ for all $G \in \mathfrak{F}$. The set $\mathcal{L}_\infty$ of all $\tau$-closed totally saturated formations is a complete lattice [3].

A $\tau$-closed totally saturated formation $\mathfrak{F}$ is called $\mathfrak{F}^\tau_\infty$-critical (or a minimal $\tau$-closed totally saturated non-$\mathfrak{H}$-formation) if $\mathfrak{F} \not\subseteq \mathfrak{H}$ but all proper $\tau$-closed totally saturated subformations of $\mathfrak{F}$ are contained in $\mathfrak{H}$.

If $\mathfrak{F}$ and $\mathfrak{M}$ are $\mathcal{L}_\infty$-formations such that $\mathfrak{M} \subseteq \mathfrak{F}$, then $\mathfrak{F}/\mathcal{L}_\infty\mathfrak{M}$ denotes the lattice of $\mathcal{L}_\infty$-formations between $\mathfrak{M}$ and $\mathfrak{F}$. In particular, if $\mathfrak{M} = (1)$ is the formation of identity groups, then $L^\tau_\infty(\mathfrak{F})$ denotes the lattice $\mathfrak{F}/\mathcal{L}_\infty(1)$.

In this paper we prove the following.

**Theorem 1.** Let $\mathfrak{F}$ and $\mathfrak{X}$ be $\tau$-closed totally saturated formations, $\mathfrak{F} \not\subseteq \mathfrak{X} \subseteq \mathfrak{N}$. Then the following conditions are equivalent:

1) the lattice $\mathfrak{F}/\mathcal{L}_\infty\mathfrak{X} \cap \mathfrak{X}$ is Boolean;
2) $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{X}) \lor \mathcal{L}_\infty(\mathfrak{H}_i | i \in I)$, where $\{\mathfrak{H}_i | i \in I\}$ is the set of all $\mathfrak{X}_\infty$-critical subformations of $\mathfrak{F}$;
3) every subformation of $\mathfrak{F}$ of the form $(\mathfrak{F} \cap \mathfrak{X}) \lor \mathcal{L}_\infty \mathfrak{H}$ is $\mathcal{L}_\infty$-complemented in $\mathfrak{F}/\mathcal{L}_\infty\mathfrak{F} \cap \mathfrak{X}$, where $\mathfrak{H}$ is some $\mathfrak{X}_\infty$-critical subformation of $\mathfrak{F}$;
4) any $\mathfrak{X}_\infty$-critical subformation of $\mathfrak{F}$ has an $\mathfrak{X}_\infty$-complement in $\mathfrak{F}$.

Note that if in this theorem $\mathfrak{X} = \mathfrak{N}$ and $\tau$ is the trivial subgroup functor (i.e., $\tau(G) = \{G\}$ for all groups $G$) we obtain the main result in [6]. In another special case ($\mathfrak{X} = (1)$ and $\mathfrak{F}$ is soluble) we obtain the main result of Section 4.3 in [3]. In particular, we give a solution of Question 4.3.16 in [3].

1. Definitions and Notations

Let $\mathfrak{X}$ be a set of groups. Then $\mathcal{L}_\infty\text{form}\mathfrak{X}$ is the $\tau$-closed totally saturated formation generated by $\mathfrak{X}$, i.e., $\mathcal{L}_\infty\text{form}\mathfrak{X}$ is the intersection of all $\tau$-closed totally saturated formations containing $\mathfrak{X}$. If $\mathfrak{X} = \{G\}$, then the formation $\mathcal{L}_\infty\text{form}G$ is called a one-generated $\tau$-closed totally saturated formation.

We denote by $\pi(\mathfrak{F})$ the set of prime divisors of orders of groups in $\mathfrak{F}$.

For any two $\tau$-closed totally saturated formations $\mathfrak{M}$ and $\mathfrak{H}$, we write $\mathfrak{M} \lor \mathcal{L}_\infty \mathfrak{H} = \mathcal{L}_\infty\text{form}(\mathfrak{M} \cup \mathfrak{H})$.

For any set $\mathfrak{X}$ of groups, we put $\mathfrak{X}_\infty(p) = \mathcal{L}_\infty\text{form}(G/F_p(G)|G \in \mathfrak{X})$, if $p \in \pi(\mathfrak{X})$, and $\mathfrak{X}_\infty(p) = \emptyset$ if $p \notin \pi(\mathfrak{X})$.

If $\mathfrak{F}$ is an arbitrary $\tau$-closed totally saturated formation, then the symbol $\mathfrak{F}_\infty$ denotes the minimal $\mathcal{L}_\infty$-valued local satellite of $\mathfrak{F}$.

For an arbitrary sequence of primes $p_1, p_2, \ldots, p_n$ and any set $\mathfrak{X}$ of groups, the class of groups $\mathfrak{X}^{p_1p_2\ldots p_n}$ is defined as follows:

1) $\mathfrak{X}^{p_1} = (A/F_{p_1}(A)|A \in \mathfrak{X})$;
2) \( \mathcal{X}^{p_1p_2...p_n} = (A/F_n(A))|A \in \mathcal{X}^{p_1p_2...p_{n-1}} \).

A sequence of primes \( p_1, p_2, \ldots, p_n \) is called suitable for \( \mathcal{X} \) if \( p_1 \in \pi(\mathcal{X}) \) and for any \( i \in \{2, \ldots, n\} \) we have \( p_i \in \pi(\mathcal{X}^{p_1p_2...p_{i-1}}) \).

Let \( p_1, p_2, \ldots, p_n \) be a suitable sequence for \( \mathcal{F} \). Then the \( l_\infty^\tau \)-valued local satellite \( \mathcal{F}_\infty^\tau p_1p_2...p_n \) is defined as follows:

1) \( \mathcal{F}_\infty^\tau p_1 = (\mathcal{F}_\infty^\tau(p_1))_\infty \);

2) \( \mathcal{F}_\infty^\tau p_1 ... p_n = (\mathcal{F}_\infty^\tau p_1 ... p_{n-1}(p_n))_\infty \).

A group \( G \) is called a \( \tau \)-minimal non-\( \mathcal{H} \)-group (or an \( \mathcal{H} \)-critical group) if \( G \not\in \mathcal{H} \) but every proper \( \tau \)-subgroup of \( G \) belongs to \( \mathcal{H} \).

A \( \tau \)-closed totally saturated formation \( \mathcal{F} \) is called an \( l_\infty^\tau \)-irreducible formation if \( \mathcal{F} \neq l_\infty^\tau \text{form}(\bigcup_{i \in I} \mathcal{X}_i) = \vee_\infty^\tau (\mathcal{X}_i | i \in I) \), where \( \mathcal{X}_i | i \in I \) is the set of all proper \( \tau \)-closed totally saturated subformations of \( \mathcal{F} \). Otherwise, \( \mathcal{F} \) is called an \( l_\infty^\tau \)-reducible \( \tau \)-closed totally saturated formation.

Let \( \mathcal{M} \) and \( \mathcal{H} \) be some \( \tau \)-closed totally saturated subformations of \( \mathcal{F} \), \( \mathcal{X} \) be a class of groups. Then \( \mathcal{H} \) is called an \( \mathcal{X}_\infty^\tau \)-complement to \( \mathcal{M} \) in \( \mathcal{F} \) if \( \mathcal{F} = l_\infty^\tau \text{form}(\mathcal{M} \cup \mathcal{H}) \) and \( \mathcal{M} \cap \mathcal{H} \subseteq \mathcal{X} \). A subformation of \( \mathcal{F} \) is called \( \mathcal{X}_\infty^\tau \)-complemented in \( \mathcal{F} \) if it has an \( \mathcal{X}_\infty^\tau \)-complement in \( \mathcal{F} \). In addition, the \( (1)_\infty^\tau \)-complement to \( \mathcal{M} \) in \( \mathcal{F} \) is called an \( l_\infty^\tau \)-complement to \( \mathcal{M} \) in \( \mathcal{F} \), and in this case \( \mathcal{M} \) is called \( l_\infty^\tau \)-complemented in \( \mathcal{F} \). A subformation \( \mathcal{M} \) of \( \mathcal{F} \) is called complemented in \( \mathcal{F} \) if \( \mathcal{F} = \text{form}(\mathcal{M} \cup \mathcal{H}) \) and \( \mathcal{M} \cap \mathcal{H} = (1) \) for some subformation \( \mathcal{H} \) of \( \mathcal{F} \).

For a set \( \pi \) of primes, we use \( \mathcal{M}_\pi \) and \( \mathcal{G}_\pi \) to denote the class of all nilpotent \( \pi \)-groups and the class of all soluble \( \pi \)-groups, respectively.

2. Used Results

**Lemma 1.** [7, 8]. Let \( \mathcal{F} \) be a non-soluble \( \tau \)-closed totally saturated formation. Then \( \mathcal{F} \) has at least one \( \mathcal{G}_\infty^\tau \)-critical subformation.

**Lemma 2.** [7, 8]. Let \( \mathcal{F} \) be a \( \tau \)-closed totally saturated formation. Then \( \mathcal{F} \) is a minimal \( \tau \)-closed totally saturated non-soluble formation if and only if \( \mathcal{F} = l_\infty^\tau \text{form}G \), where \( G \) is a monolithic \( \tau \)-minimal non-soluble group with a non-abelian minimal normal subgroup \( R \) such that \( G/R \) is soluble.

**Lemma 3.** [7, 8]. Let \( G \) be a monolithic group with a non-abelian socle \( R \). Then \( \mathcal{F} = l_\infty^\tau \text{form}G \) has a unique maximal \( l_\infty^\tau \)-subformation \( \mathcal{M} = \mathcal{G}_\pi(R)l_\infty^\tau \text{form}(\{G/R\} \cup \mathcal{X}) \), where \( \mathcal{X} \) is the set of all proper \( \tau \)-subgroups of \( G \). In particular, \( \mathcal{G}_\pi(R) \subseteq \mathcal{M} \subseteq \mathcal{F} \).

**Lemma 4.** [9]. The lattice \( l_\infty^\tau \) of \( \tau \)-closed totally saturated formations is distributive.
Lemma 5. [10]. For any two $\tau$-closed totally saturated formations $M$ and $\mathcal{F}$ we have

$$M \vee^\tau \mathcal{F} / \mathcal{F} \simeq \mathcal{F} / \mathcal{F} \cap M.$$ 

Lemma 6. [7, 11] The lattice $l^\tau_{\infty}$ is algebraic.

3. Main Results

Lemma 7. [7]. Let $\mathcal{F}$, $X$ be $\tau$-closed totally saturated formations such that $\mathcal{F} \not\subseteq X \subseteq \mathcal{N}$. The formation $\mathcal{F}$ is an $X_{\infty}$-critical formation if and only if either of the following conditions is satisfied:

1) $\mathcal{F} = \mathcal{N}_p$, where $p \not\in \pi(X)$;

2) $\mathcal{F} = \mathcal{N}_p \mathcal{N}_q$ for some different primes $p$ and $q$ in $\pi(X)$.

Proof. Necessity. Let $\mathcal{F}$ be an $X_{\infty}$-critical formation. Suppose that there exists $p \in \pi(\mathcal{F})$ such that $p \not\in \pi(X)$. Since $\mathcal{N}_p \in l^\tau_{\infty}$, $\mathcal{N}_p \subseteq \mathcal{F} \setminus X$, (1) is the unique $l^\tau_{\infty}$-subformation of $\mathcal{N}_p$ and $1) \subseteq X$, we have that $\mathcal{F} = \mathcal{N}_p$. So, $\mathcal{F}$ satisfies 1).

Assume that $\pi(\mathcal{F}) \subseteq \pi(X)$. We show that $\mathcal{F}$ is soluble.

Assume that $\mathcal{F} \not\subseteq \mathcal{S}$. Then by Lemma 1, $\mathcal{F}$ contains at least one $\mathcal{S}_{\infty}$-critical subformation $L$. By Lemma 2, $L = l^\tau_{\infty}$form$L$, where $L$ is a monolithic $\tau$-minimal non-soluble group with a non-abelian minimal normal subgroup $N$ such that group $L/N$ is soluble. It follows from Lemma 3 that $\mathcal{S}_x \subseteq \mathcal{L}$, where $\pi = \pi(N)$. Since $N$ is non-abelian, we have that $|\pi| \geq 3$. But by hypothesis the formation $\mathcal{F}$ is an $X_{\infty}$-critical formation. Hence $\mathcal{S}_x \subseteq X \subseteq \mathcal{N}$, a contradiction. Therefore, $\mathcal{F}$ is soluble.

Let $h$ be the canonical local satellite of $X$. By Theorem 2.5.2 [3, p. 94], $\mathcal{F} = l^\tau_{\infty}$form$G$, where $G$ is a group of minimal order in $\mathcal{F} \setminus X$ with the socle $R = G^x$ such that for all $p \in \pi(R)$ the formation $\mathcal{F}^\tau_{\infty}(p)$ is $(h(p))_{\infty}$-critical. Since by Theorem 1.3.14 [3, p. 33] $\mathcal{M}_\infty^\tau(p) = (1)$, we have $h(p) = \mathcal{N}_p$. Hence, $\mathcal{F}^\tau_{\infty}(p) = l^\tau_{\infty}$form$(G/F_p(G))$ is an $(\mathcal{N}_p)_{\infty}$-critical formation. Therefore, $|\pi(\mathcal{F}^\tau_{\infty}(p))| = 1$ and $\mathcal{F}^\tau_{\infty}(p) = \mathcal{N}_q$, for some prime $q \not= p$. Since $G$ is soluble, it follows that $R$ is a $p$-group and $F_p(G) = R$. Hence, $\pi(G) = \{p, q\}$ and $\mathcal{F} = \mathcal{N}_p \mathcal{N}_q$. Thus, $\mathcal{F}$ satisfies 2).

Sufficiency. Let $\mathcal{F}$ be a formation satisfying 1) or 2). Then $\mathcal{F}$ is a hereditarily totally saturated formation. Hence, $\mathcal{F}$ is a $\tau$-closed formation, for any subgroup functor $\tau$. If $\mathcal{F} = \mathcal{N}_p$, then $1)$ is a unique maximal $l^\tau_{\infty}$-subformation of $\mathcal{F}$. But $1) \subseteq X \not= \emptyset$. Hence, $\mathcal{F}$ is an $X_{\infty}$-critical formation.

Let $\mathcal{F} = \mathcal{N}_p \mathcal{N}_q$. Then by Theorem 2.5.3. [3, p. 94] $\mathfrak{m}athfrakF$ is an $\mathcal{N}_\infty^\tau$-critical formation. Since $\mathcal{N}_{\{p,q\}} \subseteq X$, it follows that $\mathcal{F}$ is a minimal $\tau$-closed totally saturated non-$X$-formation. 

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Lemma 8. [7]. Let \( \mathcal{F} \) and \( \mathcal{X} \) be \( l^r_\infty \)-formations such that \( \mathcal{F} \subseteq \mathcal{X} \subseteq \mathcal{N} \). Then \( \mathcal{F} \) has at least one \( \mathcal{X}^r_\infty \)-critical subformation.

Proof. Assume that \( \pi(\mathcal{F}) \not\subseteq \pi(\mathcal{X}) \) and \( p \in \pi(\mathcal{F}) \setminus \pi(\mathcal{X}) \). Then according to Lemma 6, \( \mathcal{N}_p \) is a required \( \mathcal{X}^r_\infty \)-critical formation. Now we assume that \( \pi(\mathcal{F}) \subseteq \pi(\mathcal{X}) \), and let \( A \) be a group of minimal order in \( \mathcal{F} \setminus \mathcal{X} \). Then \( A \) is a monolithic \( \tau \)-minimal non-\( \mathcal{X} \)-group with the socle \( R = A^\mathcal{X} \). Let \( p \in \pi(R) \) and \( \mathcal{L} = l^*_\infty \text{form}A \). Assume that \( R \) is non-abelian. Then by Lemma 3, \( \mathcal{G}_{\pi(R)} \subseteq \mathcal{L} \). Since \( |\pi(R)| \geq 3 \), there exists a prime \( q \neq p \), \( q \in \pi(R) \), such that

\[
\mathcal{M} = \mathcal{N}_p \mathcal{N}_q \subset \mathcal{G}_{\pi(R)} \subset \mathcal{F}.
\]

Since \( \mathcal{N}_{\{p,q\}} \subseteq \mathcal{X} \), from Lemma 6 it follows that \( \mathcal{M} \) is a required \( \mathcal{X}^r_\infty \)-critical formation.

Suppose now that \( R \) is an abelian \( p \)-group. Since \( R \not\subseteq \Phi(A) \), we have \( R = O_p(A) = F_p(A) \) and \( A = [R]B \) for some maximal subgroup \( B \) in \( A \). By Theorem 1.3.14 [3, p. 33],

\[
\mathcal{L}^r_\infty (p) = l^*_\infty \text{form}(A/F_p(A)) = l^*_\infty \text{form}B.
\]

Let \( q \in \pi(B) \setminus \{p\} \), and \( Q \) be a group of prime order \( q \). Since \( \mathcal{L}^r_\infty (p) \) is totally saturated, \( Q \in \mathcal{L}^r_\infty (p) \). Denote by \( V \) an exact irreducible \( F_p[Q] \)-modul, and let \( F = [V]Q \). Then

\[
F/O_p(F) \simeq Q \in \mathcal{L}^r_\infty (p).
\]

Therefore, by Lemma 8.2 [2, p. 78], \( F \in \mathcal{L} \). But

\[
\mathcal{F} = l^*_\infty \text{form}F = \mathcal{N}_p \mathcal{N}_q.
\]

Hence, by Lemma 6, \( \mathcal{F} \) is a required \( \mathcal{X}^r_\infty \)-critical formation. \( \square \)

Lemma 9. Let \( \mathcal{X} \), \( \mathcal{M} \) and \( \mathcal{F} \) be \( \tau \)-closed totally saturated formations such that \( \mathcal{M} \subseteq \mathcal{X} \subseteq \mathcal{N} \), and \( \mathcal{F} = \mathcal{M} \vee_{\mathcal{X}} (\vee_{\mathcal{N}}(\mathcal{H}_i|i \in I)) \), where \( \{\mathcal{H}_i|i \in I\} \) is some set of \( \mathcal{X}^r_\infty \)-critical formations. If \( \mathcal{H} \) is an \( \mathcal{X}^r_\infty \)-critical subformation of \( \mathcal{F} \), then \( \mathcal{H} \in \{\mathcal{H}_i|i \in I\} \).

Proof. Let \( \mathcal{H} \) be a \( \mathcal{X}^r_\infty \)-critical subformation of \( \mathcal{F} \). By Lemma 6, \( \mathcal{H} \) satisfies either of the following conditions:

1) \( \mathcal{H} = \mathcal{N}_p \), where \( p \notin \pi(\mathcal{X}) \);

2) \( \mathcal{H} = \mathcal{N}_p \mathcal{N}_q \) for some primes \( p \neq q \) in \( \pi(\mathcal{X}) \).

Assume that \( \mathcal{H} \) satisfies 1). Since \( \mathcal{H} \subseteq \mathcal{F} \), we have by Corollary 1.3.10 [3, p. 31] that \( \mathcal{H}^r_\infty \subseteq \mathcal{F}^r_\infty \). Therefore, \( \mathcal{H}^r_\infty (p) \subseteq \mathcal{F}^r_\infty (p) \). By Theorem 1.3.14 [3, p. 33], we have \( \mathcal{H}^r_\infty (p) = (1) \). Hence, \( (1) \subseteq \mathcal{F}^r_\infty (p) = \emptyset \).

By Lemma 4.1.2 [3, p. 152],

\[
\mathcal{F}^r_\infty (p) = \mathcal{M}^r_\infty (p) \vee_{\mathcal{X}} (\vee_{\mathcal{N}}(\mathcal{H}^r_\infty (p)|i \in I)).
\]
Since \( p \notin \pi(X) \), it follows that \( p \notin \pi(M) \) and \( M^{-\infty}(p) = \emptyset \). Hence,
\[
\mathcal{F}_\infty^-(p) = \bigvee_\infty (\mathcal{F}_\infty^-(p) | i \in I).
\]

Suppose that \( p \notin \pi(H_i) \) for all \( i \in I \). Then from Theorem 1.3.14 [3, p. 33] it follows that \( \mathcal{F}_\infty^-(p) = \emptyset \) for all \( i \in I \). Therefore, \( \mathcal{F}_\infty^-(p) = \emptyset \), a contradiction. So, there exists \( i \in I \) such that \( p \in \pi(H_i) \). Since \( H_i \) is an \( \mathcal{X}_\infty^\tau \)-critical formation and \( p \notin \pi(X) \), we see that \( H_i = N_p \). Thus, \( H_i = H \).

Assume that \( H \) satisfies 2). Then \( p, q \) is a suitable sequence for \( H \) and \( \mathcal{F} \). By Corollary 1.3.10 and Theorem 1.3.14 [3], we obtain that
\[
H^{-\infty}(p) \subseteq \mathcal{F}^{-\infty}(p) \quad \text{and} \quad H^{-\infty}(p) = (1) \subseteq \mathcal{F}^{-\infty}(p) \neq \emptyset.
\]

From Lemma 4.1.2 [3, p. 152] it follows that
\[
\mathcal{F}^{-\infty}(p) = M^{-\infty}(p) \vee_\infty (\bigvee_\infty (H^{-\infty}(p) | i \in I)).
\]

Suppose that \( q \in \pi(M^{-\infty}(p)) \). Since \( M^{-\infty}(p) \) is a saturated formation, we have that \( N_q \subseteq M^{-\infty}(p) \). By Theorem 1.3.12 [3, p. 32],
\[
N_pM^{-\infty}(p) \subseteq M.
\]
Hence,
\[
H = N_pN_q \subseteq N_pM^{-\infty}(p) \subseteq M \subseteq X.
\]

But \( H \) is an \( \mathcal{X}_\infty^\tau \)-critical formation. We have a contradiction. Therefore, \( q \notin \pi(M^{-\infty}(p)) \), \( M^{-\infty}(p) = \emptyset \) and
\[
\mathcal{F}^{-\infty}(p) = (\bigvee_\infty (H^{-\infty}(p) | i \in I)).
\]

If \( H^{-\infty}(p) = \emptyset \) for all \( i \in I \), then \( \mathcal{F}^{-\infty}(p) = \emptyset \). It is impossible. Therefore, there exists \( i \in I \) such that \( H^{-\infty}(p) \neq \emptyset \). Hence, \( q \in \pi(H^{-\infty}(p)) \) and \( N_q \subseteq H^{-\infty}(p) \). But by Theorem 1.3.12 [3] we have \( N_pH^{-\infty}(p) \subseteq H \). Therefore,
\[
H = N_pN_q \subseteq N_pH^{-\infty}(p) \subseteq H.
\]
Since \( H \) is an \( \mathcal{X}_\infty^\tau \)-critical formation, we see that \( H = H \).

\[\square\]

**Lemma 10.** \( \ast \) Let \( X, M, L \), and \( \mathcal{F} \) be \( \tau \)-closed totally saturated formations such that \( X \subseteq M \subseteq L \subseteq \mathcal{F} \). If \( H \) is an \( L^{-\infty} \)-complement to \( M \) in \( \mathcal{F}/\tau X \), then \( H \cap L \) is an \( L^{-\infty} \)-complement to \( M \) in \( L/\tau X \).

**Proof.** Let \( H_1 = H \cap L \). Since \( M \) is \( L^{-\infty} \)-complemented in the lattice \( \mathcal{F}/\tau X \) by \( H \), it follows that \( M \cap H = X \) and \( M \cap H = \mathcal{F} \). From Lemma 4 it follows that
\[
M \vee_\infty H_1 = M \vee_\infty (H \cap L) = (M \vee_\infty H) \cap (M \vee_\infty L) = \mathcal{F} \cap L = L.
\]
Besides,
\[ M \cap H_1 = M \cap (H \cap L) = M \cap H = X. \]

But then \( H_1 \) is an \( \ell_\infty^\tau \)-complement to \( M \) in \( L/\ell_\infty^\tau X \).

**Lemma 11.** Let \( X \) and \( F \) be \( \tau \)-closed totally saturated formations, \( H \) be some \( X_\infty^\tau \)-critical subformation of \( F \). Then \( H \) has an \( X_\infty^\tau \)-complement in \( F \) if and only if \( H \vee_\infty^\tau (F \cap X) \) has an \( \ell_\infty^\tau \)-complement in \( F/\ell_\infty^\tau F \cap X \).

**Proof.** Let \( M \) be an \( X_\infty^\tau \)-complement to \( H \) in \( F \). Then by definition \( H \cap M \subseteq X \) and \( H \vee_\infty^\tau M = F \). Put \( M_1 = M \vee_\infty^\tau (F \cap X) \) and \( H_1 = H \vee_\infty^\tau (F \cap X) \). Then \( M_1 \) and \( H_1 \) are elements of the lattice \( F/\ell_\infty^\tau F \cap X \). By Lemma 4,
\[ H_1 \cap M_1 = H_1 \cap (M \vee_\infty^\tau (F \cap X)) = (H_1 \cap M) \vee_\infty^\tau (H_1 \cap (F \cap X)) = (H \vee_\infty^\tau (F \cap X)) \cap M \vee_\infty^\tau (F \cap X) = (H \cap M) \vee_\infty^\tau (M \cap X) \vee_\infty^\tau (F \cap X) = F \cap X. \]

Besides,
\[ H_1 \vee_\infty^\tau M_1 = H_1 \vee_\infty^\tau (F \cap X) \vee_\infty^\tau M \vee_\infty^\tau (F \cap X) = F. \]

Therefore, \( M_1 \) is an \( \ell_\infty^\tau \)-complement to \( H_1 \) in the lattice \( F/\ell_\infty^\tau F \cap X \).

Conversely, assume that \( H_1 \) has an \( \ell_\infty^\tau \)-complement \( M \) in the lattice \( F/\ell_\infty^\tau F \cap X \). Then \( H_1 \cap M = F \cap X \) and \( H_1 \vee_\infty^\tau M = F \). Hence, by definition, \( M \) is an \( X_\infty^\tau \)-complement to \( H_1 \) in \( F \).

**Proof of Theorem 1.** For an arbitrary \( \ell_\infty^\tau \)-formation \( L \), we denote by \( \Omega(L) \) the set of all its \( X_\infty^\tau \)-critical subformations.

Assume that for \( \mathcal{F} \) Condition 1) is true, and \( M = (F \cap X) \vee_\infty^\tau (H \cap X) \in \Omega(F) \). Assume that \( M \neq \mathcal{F} \). Since \( \mathcal{F} \cap X \subseteq M \subseteq \mathcal{F} \), \( M \) is an element of the lattice \( F/\ell_\infty^\tau F \cap X \). Let \( L \) be an \( \ell_\infty^\tau \)-complement to \( M \) in the lattice \( F/\ell_\infty^\tau F \cap X \). Then \( M \vee_\infty^\tau L = F \) and \( M \cap L = \mathcal{F} \cap X \). If \( L \subseteq X \), then \( L \subseteq F \cap X \subseteq M \) and \( F = M \vee_\infty^\tau L = M \), which contradicts to our assumption. Therefore, \( L \not\subseteq X \). Hence, by Lemma 8, the formation \( L \) contains at least one \( X_\infty^\tau \)-critical subformation \( H \). Since \( H \subseteq L \subseteq F \), we have that \( H \in \Omega(F) \subseteq M \). But then \( H \subseteq L \cap M = F \cap X \), a contradiction. Hence, \( M = \mathcal{F} \).

Now we show that Condition 2) implies Condition 3). Let \( H_1 \) be an \( X_\infty^\tau \)-critical subformation of the formation \( F \), \( \Sigma = \Omega(F) \setminus \{ H_1 \} \),
\[ L = (F \cap X) \vee_\infty^\tau H_1 \quad \text{and} \quad M = (F \cap X) \vee_\infty^\tau (H \cap X) \in \Sigma. \]

Then \( L \vee_\infty^\tau M = F \). Suppose that \( L \cap M \neq F \cap X \). Since \( \mathcal{F} \cap X \subseteq L \cap M \), we have \( L \cap M \not\subseteq F \cap X \), i.e., \( L \cap M \not\subseteq X \). Then by Lemma 8, \( L \cap M \) contains some \( X_\infty^\tau \)-critical subformation \( H_2 \). Since \( H_2 \subseteq L \), it follows
from Lemma 9 that \( h_2 = S_1 \). But \( h_2 \subseteq M \). Hence by Lemma 9, \( h_2 \in \Sigma \), a contradiction. Thus, \( L \cap M = \mathfrak{F} \cap X \). It means that the formation \( L \) is \( \mathfrak{F} \)-complemented in the lattice \( \mathfrak{F}/\mathfrak{F} \cap X \). So, Condition 3) is true for \( \mathfrak{F} \).

Now we assume that for \( \mathfrak{F} \) Condition 3) is true. We show that Condition 1) is true. By Lemma 4, the lattice \( \mathfrak{F}/\mathfrak{F} \cap X \) is distributive. Therefore, it is enough to establish that \( \mathfrak{F} \cap X \) is a complemented lattice.

Let \( M \) be an \( \mathfrak{F} \)-irreducible \( \tau \)-closed totally saturated subformation of \( \mathfrak{F} \), \( M \not\subseteq X \). We prove that \( M \) is an \( \mathfrak{X}_\tau \)-critical formation. Suppose that it is false, and let \( M_1 \) be a maximal \( \mathfrak{F} \)-subformation in \( M \). Since \( M \) is non-\( \mathfrak{X}_\tau \)-critical, \( M_1 \not\subseteq X \). Hence, by Lemma 8 the formation \( M_1 \) has at least one \( \mathfrak{X}_\tau \)-critical subformation \( h \). Let \( L = H \cap X \). Then \( L \) is an element of the lattice \( \mathfrak{F}/\mathfrak{F} \cap X \). Let \( M \) be an \( \mathfrak{F} \)-complement to \( L \) in \( \mathfrak{F}/\mathfrak{F} \cap X \). Then \( M = R \cap \mathfrak{F} \cap X \) and \( L \cap M = L \cap X \). By Lemma 11, \( \mathfrak{F} \cap X \) is an \( \mathfrak{F} \)-complement to \( L \) in the lattice \( \mathfrak{F} \cap X \). Therefore,

\[
(M \cap (M \cap X)) \cap X = M \cap X.
\]

By Lemma 4,

\[
M \cap (M \cap X) = (M \cap M) \cap X.
\]

It means that

\[
M \cap (M \cap X) \subseteq M_1 \cap X.
\]

Since \( L \subseteq M_1 \cap X \) we have that

\[
(M \cap (M \cap X)) \cap X \subseteq M_1 \cap X.
\]

But \( (M \cap X) \cap X = M \cap X \). Hence,

\[
M \cap X \subseteq M_1 \cap X.
\]

The inverse inclusion is obvious. Therefore,

\[
M \cap X = M_1 \cap X.
\]

But by Lemma 5 we have a lattice isomorphism

\[
M \cap X \cong M_1 \cap X.
\]
Therefore, $\mathcal{M}_1 \vee^\tau_\infty (\mathcal{F} \cap \mathcal{X})$ is a maximal $\tau$-closed totally saturated subformation of the formation $\mathcal{M} \vee^\tau_\infty (\mathcal{F} \cap \mathcal{X})$. We obtain a contradiction. Hence, $\mathcal{M}$ is an $\mathcal{X}_\infty^\tau$-critical formation.

We show now that for any $l^\tau_\infty$-formation $\mathcal{R}$ in $\mathcal{F}/\mathcal{F} \cap \mathcal{X}$ such that the set of all its $\mathcal{X}_\infty^\tau$-critical subformations is finite, the following equality is true:

$$\mathcal{R} = (\mathcal{F} \cap \mathcal{X}) \vee^\tau_\infty (\vee^\tau_\infty (\mathcal{H} | \mathcal{H} \in \Omega(\mathcal{R}))).$$

We shall prove $(\alpha)$ by induction on $|\Omega(\mathcal{R})|$. If $\mathcal{R}$ is an $l^\tau_\infty$-irreducible formation, then from above we know that $\mathcal{R}$ is a $\mathcal{X}_\infty^\tau$-critical formation, and $(\alpha)$ is true. Let $\mathcal{R}$ be an $l^\tau_\infty$-reducible formation. Since $\mathcal{R} \not\subseteq \mathcal{X}$, we have by Lemma 8 that $\mathcal{R}$ contains some $\mathcal{X}_\infty^\tau$-critical formation $\mathcal{H}_1$. Let $\mathcal{H}_1 = \mathcal{H} \vee^\tau_\infty (\mathcal{F} \cap \mathcal{X})$. By hypothesis, $\mathcal{H}_1$ has an $l^\tau_\infty$-complement $\mathcal{M}$ in the lattice $\mathcal{F}/\mathcal{F} \cap \mathcal{X}$.

By Lemma 11, $\mathcal{M} \cap \mathcal{R}$ is a complement to $\mathcal{H}_1$ in the lattice $\mathcal{R}/\mathcal{F} \cap \mathcal{X}$. Then

$$(\mathcal{M} \cap \mathcal{R}) \cap \mathcal{H}_1 = \mathcal{F} \cap \mathcal{X} \text{ and } (\mathcal{M} \cap \mathcal{R}) \vee^\tau_\infty \mathcal{H}_1 = \mathcal{R}.$$ 

Since $\mathcal{H} \not\subseteq \mathcal{M}$, the number of $\mathcal{X}_\infty^\tau$-critical subformations of $\mathcal{M} \cap \mathcal{R}$ is less than the number of $\mathcal{X}_\infty^\tau$-critical subformations in $\mathcal{R}$. Therefore, by induction we can conclude that

$$\mathcal{M} \cap \mathcal{R} = (\mathcal{F} \cap \mathcal{X}) \vee^\tau_\infty (\vee^\tau_\infty (\mathcal{B} | \mathcal{B} \in \Omega(\mathcal{M} \cap \mathcal{R}))).$$

Hence,

$$\mathcal{R} = (\mathcal{M} \cap \mathcal{R}) \vee^\tau_\infty \mathcal{H}_1 =
= ((\mathcal{F} \cap \mathcal{X}) \vee^\tau_\infty (\vee^\tau_\infty (\mathcal{B} | \mathcal{B} \in \Omega(\mathcal{M} \cap \mathcal{R})))) \vee^\tau_\infty (\mathcal{H} \vee^\tau_\infty (\mathcal{F} \cap \mathcal{X})) =
= (\mathcal{F} \cap \mathcal{X}) \vee^\tau_\infty (\vee^\tau_\infty (\mathcal{B} | \mathcal{B} \in \Omega(\mathcal{R}))),$$

i.e., $(\alpha)$ is true.

Let now $\mathcal{M}$ be an $l^\tau_\infty$-subformation of $\mathcal{F}/\mathcal{F} \cap \mathcal{X}$. Assume that

$$\mathcal{L} = (\mathcal{F} \cap \mathcal{X}) \vee^\tau_\infty (\vee^\tau_\infty (\mathcal{H} | \mathcal{H} \in \Omega(\mathcal{F} \setminus \Omega(\mathcal{M}))).$$

We show that $\mathcal{L}$ is an $l^\tau_\infty$-complement to $\mathcal{M}$ in the lattice $\mathcal{F}/\mathcal{F} \cap \mathcal{X}$.

It is obvious that $\mathcal{F} \cap \mathcal{X} \subseteq \mathcal{M} \cap \mathcal{L}$. If $\mathcal{M} \cap \mathcal{L} \not\subseteq \mathcal{F} \cap \mathcal{X}$, then by Lemma 8, $\mathcal{M} \cap \mathcal{L}$ has at least one $\mathcal{X}_\infty^\tau$-critical subformation $\mathcal{H}_1$. But then, using Lemma 9, we have that $\mathcal{H}_1 \in \Omega(\mathcal{M}) \cap (\Omega(\mathcal{F} \setminus \Omega(\mathcal{M})) = \emptyset$, a contradiction. Hence, $\mathcal{M} \cap \mathcal{L} = \mathcal{F} \cap \mathcal{X}$.

Let $\mathcal{F}_1 = \mathcal{L} \vee^\tau_\infty \mathcal{M}$. Suppose that $\mathcal{F}_1 \not= \mathcal{F}$ and $G$ is a group in $\mathcal{F} \setminus \mathcal{F}_1$.

Since $\pi(G)$ is a finite set, by Lemma 7 the set of all $\mathcal{X}_\infty^\tau$-critical subformations of the formation $\mathcal{R} = l^\tau_\infty\text{form}G$ is finite. Denote by $\mathcal{R}_1$ the formation $\mathcal{R} \vee^\tau_\infty (\mathcal{F} \cap \mathcal{X})$. By Lemma 9, the set of all $\mathcal{X}_\infty^\tau$-critical
subformations of the formation $\mathfrak{R}_1$ is finite. Therefore, by $(\alpha)$ we have that
\[ \mathfrak{R}_1 = (\mathfrak{R} \cap \mathfrak{X}) \vee_{\infty} (\bigvee_{\infty}(\mathfrak{H}|\mathfrak{H} \in \Omega(\mathfrak{R}))) \].

Since $\Omega(\mathfrak{R}_1) \subseteq \Omega(\mathfrak{R}) = \Omega(\mathfrak{L}) \cup \Omega(\mathfrak{M})$ and $\mathfrak{R} \cap \mathfrak{X} \subseteq \mathfrak{R}_1$, it follows that $\mathfrak{R}_1 \subseteq \mathfrak{R}_1$. Therefore, $G \in \mathfrak{R}_1$, a contradiction. So, $\mathfrak{R}_1$, and $\mathfrak{R}/\infty \mathfrak{R} \cap \mathfrak{X}$ is a complemented lattice.

In particular, if $\mathfrak{X} = (1)$, from Theorem 1 we deduce the following result.

**Theorem 2.** Let $\mathfrak{F}$ be a $\tau$-closed totally saturated formation. Then the following conditions are equivalent:

1) the lattice $L_{\infty}(\mathfrak{F})$ is Boolean;

2) $\mathfrak{F} = \mathfrak{M}_{\pi}(\mathfrak{F})$;

3) every subformation of the form $\mathfrak{M}_p$ in $\mathfrak{F}$ is complemented in $\mathfrak{F}$.

**Proof.** By Lemma 7, any $(1)_{\infty}$-critical formation $\mathfrak{F}$ has a form $\mathfrak{F} = \mathfrak{M}_p$, where $p$ is a prime. Therefore by Theorem 1,
\[ \mathfrak{F} = \bigvee_{\infty}(\mathfrak{M}_p|p \in \pi(\mathfrak{F})) = \mathfrak{M}_{\pi}(\mathfrak{F}). \]

Thus, Conditions 1) and 2) are equivalent to Conditions 1) and 2) of Theorem 1.

Now we show that any subformation $\mathfrak{M}_p$ of $\mathfrak{F}$ has a complement in $\mathfrak{F}$. By Theorem 1, Condition 2) is equivalent to the following: every subformation $\mathfrak{M}_p$ of $\mathfrak{F}$ has an $L_{\infty}$-complement. Let $\mathfrak{M}$ be an $L_{\infty}$-complement to $\mathfrak{M}_p$ in $\mathfrak{F}$. Then $\mathfrak{M}_p \vee_{\infty} \mathfrak{M} = \mathfrak{F}$ and $\mathfrak{M}_p \cap \mathfrak{M} = (1)$. By Theorem 1.3.16 [3, p. 34], $\mathfrak{F} = \text{form}(\bigcup_{q \in \pi(\mathfrak{F})} \mathfrak{M}_q \mathfrak{F}_{\infty}(q))$. Since $\mathfrak{F} \subseteq \mathfrak{M}$, we have by Theorem 1.3.14 [3, p. 33] that $\mathfrak{F}_{\infty}(q) = (1)$. It means that $\mathfrak{F} = \text{form}(\bigcup_{q \in \pi(\mathfrak{F})} \mathfrak{M}_q)$. Since $\mathfrak{M}$ is contained in $\mathfrak{F}$ and is an $L_{\infty}$-formation, we have by Theorem 1.3.16 [3, p. 34] that
\[ \mathfrak{M} = \text{form}(\bigcup_{q \in \pi(\mathfrak{F})} \mathfrak{M}_q) = \mathfrak{M}_{\pi}(\mathfrak{F}) \setminus \{p\}. \]

Hence,
\[ \mathfrak{F} = \text{form}(\mathfrak{M}_p \cup (\bigcup_{q \in \pi(\mathfrak{F}) \setminus \{p\}} \mathfrak{M}_q)) = \text{form}(\mathfrak{M}_p \cup \text{form}(\bigcup_{q \in \pi(\mathfrak{F}) \setminus \{p\}} \mathfrak{M}_q)) = \text{form}(\mathfrak{M}_p \cup \mathfrak{M}). \]

Thus, $\mathfrak{M}$ is a complement to $\mathfrak{M}_p$ in $\mathfrak{F}$.

Let $\mathfrak{L}$ be a complement to $\mathfrak{M}_p$ in $\mathfrak{F}$. Then $\mathfrak{M}_p \vee \mathfrak{L} = \mathfrak{F}$ and $\mathfrak{M}_p \cap \mathfrak{L} = (1)$. We show that $\mathfrak{L}$ is an $L_{\infty}$-complement to $\mathfrak{M}_p$ in $\mathfrak{F}$. Let $\mathfrak{M} = L_{\infty}\text{form}\mathfrak{L}$. Suppose that $\mathfrak{M} \not\leq \mathfrak{L}$, and let $A$ be a group of minimal order in $\mathfrak{M} \setminus \mathfrak{L}$. Then $A$ is a monolithic group, and $R = \text{Soc}(A) = A^G$. Since $A \in \mathfrak{N}$, we conclude that $A$ is a $p$-group. If $A \neq R$, then from $A/R \in \mathfrak{L}$ we have
$\mathfrak{N}_p \cap \mathfrak{L} \neq (1)$, a contradiction. It means that $A = R$, and $A$ is a group of order $p$. By Theorem 1.1.5 [3, p. 14], $\pi(\mathfrak{M}) = \pi(\mathfrak{L})$. Therefore, $p \in \pi(\mathfrak{L})$. Since $\mathfrak{L} \subseteq \mathfrak{N}_p$, we have $\mathfrak{N}_p \cap \mathfrak{L} \neq (1)$, a contradiction. Hence, $\mathfrak{M} = \mathfrak{L}$. Thus, $\mathfrak{L}$ is an $L_\infty$-complement to $\mathfrak{N}_p$ in $\mathfrak{F}$.

Theorem 2 gives the answer to Question 4.3.16 [3, p. 178].

In the case when $\tau(G) = S(G)$ is the set of all subgroups of $G$, from Theorem 1 we have the following.

**Corollary 1.** Let $\mathfrak{F}$ be a hereditary totally saturated formation. Then the following conditions are equivalent:

1) the lattice $L_{\infty}(\mathfrak{F})$ is Boolean;
2) $\mathfrak{F} = \mathfrak{N}_{\tau}(\mathfrak{F})$;
3) every subformation of the form $\mathfrak{N}_p$ in $\mathfrak{F}$ is complemented in $\mathfrak{F}$.

If $\tau(G) = S_n(G)$ is the set of all normal subgroups of $G$, from Theorem 1 we have

**Corollary 2.** Let $\mathfrak{F}$ be a normal hereditary totally saturated formation. Then the following conditions are equivalent:

1) the lattice $L_{\infty}(\mathfrak{F})$ is Boolean;
2) $\mathfrak{F} = \mathfrak{N}_{\tau}(\mathfrak{F})$;
3) every subformation of the form $\mathfrak{N}_p$ in $\mathfrak{F}$ is complemented in $\mathfrak{F}$.

Let $\tau$ be a trivial subgroup functor. Then from Theorem 1 we obtain the following.

**Corollary 4.** Let $\mathfrak{F}$ and $\mathfrak{X}$ be totally saturated formations, $\mathfrak{F} \not\subseteq \mathfrak{X} \subseteq \mathfrak{N}$. Then the following conditions are equivalent:

1) the lattice $L_{\infty}(\mathfrak{F})$ is Boolean;
2) $\mathfrak{F} \cap \mathfrak{X}$ is Boolean;
3) every subformation of the form $\mathfrak{N}_p$ in $\mathfrak{F}$ is complemented in $\mathfrak{F}$.

**Corollary 5.** [12]. Let $\mathfrak{F}$ be a totally saturated formation. Then the following conditions are equivalent:

1) $L_{\infty}(\mathfrak{F})$ is a complemented lattice;
2) \( \mathcal{F} = \mathcal{N}_\pi(\mathcal{F}) \);
3) the lattice \( L_\infty^\tau(\mathcal{F}) \) is Boolean;
4) every subformation of the form \( \mathcal{N}_p \) in \( \mathcal{F} \) is complemented in \( \mathcal{F} \).

In the case when \( \mathcal{X} = \mathcal{N} \) from Theorem 1 we have

**Corollary 6.** Let \( \mathcal{F} \) be a non-nilpotent \( \tau \)-closed totally saturated formation. Then the following conditions are equivalent:

1) the lattice \( \mathcal{F}/\mathcal{S}_\infty^\tau \cap \mathcal{N} \) is Boolean;
2) \( \mathcal{F} = (\mathcal{F} \cap \mathcal{N}) \lor (\mathcal{S}_\infty^\tau(\mathcal{F}_i)|i \in I) \), where \( \{\mathcal{F}_i|i \in I\} \) is the set of all \( \mathcal{N}_\infty^\tau \)-critical subformations of \( \mathcal{F} \);
3) every subformation of the form \( (\mathcal{F} \cap \mathcal{F}_i) \lor \mathcal{S}_\infty^\tau \mathcal{H} \) in \( \mathcal{F} \) is \( l_\infty^\tau \)-complemented in \( \mathcal{F}/\mathcal{S}_\infty^\tau \cap \mathcal{N} \), where \( \mathcal{H} \) is some \( \mathcal{N}_\infty^\tau \)-critical subformations of \( \mathcal{F} \);
4) every subformation of the form \( \mathcal{N}_p \mathcal{N}_q \) in \( \mathcal{F} \) has an \( \mathcal{N}_\infty^\tau \)-complement in \( \mathcal{F} \).

**Corollary 7.** [6]. Let \( \mathcal{F} \) be a non-nilpotent totally saturated formation. Then the following conditions are equivalent:

1) the lattice \( \mathcal{F}/\mathcal{S}_\infty^\tau \cap \mathcal{N} \) is a complemented lattice;
2) formation \( \mathcal{F} \) is soluble, and the lattice \( \mathcal{F}/\mathcal{S}_\infty^\tau \cap \mathcal{N} \) is algebraic; furthermore, \( \mathcal{F} = (\mathcal{F} \cap \mathcal{N}) \lor (\mathcal{S}_\infty^\tau(\mathcal{F}_i)|i \in I) \), where \( \{\mathcal{F}_i|i \in I\} \) is the set of all \( \mathcal{N}_\infty \)-critical subformations in \( \mathcal{F} \);
3) the lattice \( \mathcal{F}/\mathcal{S}_\infty^\tau \cap \mathcal{N} \) is Boolean.

**Proof.** By Lemma 7, every \( \mathcal{N}_\infty \)-critical formation is soluble. Then from Condition 2) of Theorem 1 the formation \( \mathcal{F} \) is soluble. By Lemma 6, the lattice \( l_\infty^\tau \) is algebraic for every subgroup functor \( \tau \). Therefore, the lattice \( \mathcal{F}/\mathcal{S}_\infty^\tau \cap \mathcal{N} \) is also algebraic (it is a sublattice of complete algebraic lattice \( l_\infty^\tau \)). Applying Theorem 1 and Lemma 4 we conclude that Conditions 1) and 3) are equivalent. \( \Box \)

**Corollary 8.** Let \( \mathcal{F} \) and \( \mathcal{X} \) be hereditary totally saturated formations, \( \mathcal{F} \subseteq \mathcal{X} \subseteq \mathcal{N} \). Then the following conditions are equivalent:

1) the lattice \( \mathcal{F}/\mathcal{S}_\infty^\tau \cap \mathcal{X} \) is Boolean;
2) \( \mathcal{F} = (\mathcal{F} \cap \mathcal{X}) \lor (\mathcal{S}_\infty^\tau(\mathcal{F}_i)|i \in I) \), where \( \{\mathcal{F}_i|i \in I\} \) is the set of all \( \mathcal{X}_\infty^S \)-critical subformations of \( \mathcal{F} \);
3) every subformation of the form \( (\mathcal{F} \cap \mathcal{F}_i) \lor \mathcal{X}_\infty^S \mathcal{H} \) in \( \mathcal{F} \) is complemented in \( \mathcal{F}/\mathcal{S}_\infty^\tau \cap \mathcal{X} \), where \( \mathcal{H} \) is some \( \mathcal{X}_\infty^S \)-critical subformations of \( \mathcal{F} \);
4) any \( \mathcal{X}_\infty^S \)-critical subformation of \( \mathcal{F} \) has an \( \mathcal{X}_\infty^S \)-complement in \( \mathcal{F} \).

**Corollary 9.** Let \( \mathcal{F} \) and \( \mathcal{X} \) be normal hereditary totally saturated formations, \( \mathcal{F} \subseteq \mathcal{X} \subseteq \mathcal{N} \). Then the following conditions are equivalent:

1) the lattice \( \mathcal{F}/\mathcal{S}_\infty^\tau \cap \mathcal{X} \) is Boolean;
2) \( \mathcal{F} = (\mathcal{F} \cap \mathbf{X}) \vee_\infty (\bigvee_\infty \mathcal{H}_i | i \in I) \), where \( \{ \mathcal{H}_i | i \in I \} \) is the set of all \( \mathcal{X}_\infty \)-critical subformations of \( \mathcal{F} \);
3) every subformation of the form \( (\mathcal{F} \cap \mathbf{X}) \vee_\infty \mathcal{H} \) in \( \mathcal{F} \) is complemented in \( \mathcal{F}/\mathcal{S}_\infty \mathcal{F} \cap \mathbf{X} \), where \( \mathcal{H} \) is some \( \mathcal{X}_\infty \)-critical subformations of \( \mathcal{F} \);
4) any \( \mathcal{X}_\infty \)-critical subformation of \( \mathcal{F} \) has an \( \mathcal{X}_\infty \)-complement in \( \mathcal{F} \).

**Corollary 10.** Let \( \mathcal{F} \) be a non-nilpotent hereditary totally saturated formation. Then the following conditions are equivalent:
1) the lattice \( \mathcal{F}/\mathcal{S}_\infty \mathcal{F} \cap \mathbf{N} \) is Boolean;
2) \( \mathcal{F} = (\mathcal{F} \cap \mathbf{N}) \vee_\infty (\bigvee_\infty \mathcal{H}_i | i \in I) \), where \( \{ \mathcal{H}_i | i \in I \} \) is the set of all \( \mathcal{N}_\infty \)-critical subformations of \( \mathcal{F} \);
3) every subformation of the form \( (\mathcal{F} \cap \mathbf{N}) \vee_\infty \mathcal{H} \) in \( \mathcal{F} \) is complemented in \( \mathcal{F}/\mathcal{S}_\infty \mathcal{F} \cap \mathbf{N} \), where \( \mathcal{H} \) is some \( \mathcal{N}_\infty \)-critical subformations of \( \mathcal{F} \);
4) every subformation of the form \( \mathcal{N}_p \mathcal{N}_q \) in \( \mathcal{F} \) has an \( \mathcal{N}_\infty \)-complement in \( \mathcal{F} \).

**Corollary 11.** Let \( \mathcal{F} \) be a non-nilpotent normal hereditary totally saturated formation. Then the following conditions are equivalent:
1) the lattice \( \mathcal{F}/\mathcal{S}_n \mathcal{F} \cap \mathbf{N} \) is Boolean;
2) \( \mathcal{F} = (\mathcal{F} \cap \mathbf{N}) \vee_\infty (\bigvee_\infty \mathcal{H}_i | i \in I) \), where \( \{ \mathcal{H}_i | i \in I \} \) is the set of all \( \mathcal{N}_\infty \)-critical subformations of \( \mathcal{F} \);
3) every subformation of the form \( (\mathcal{F} \cap \mathbf{N}) \vee_\infty \mathcal{H} \) in \( \mathcal{F} \) is complemented in \( \mathcal{F}/\mathcal{S}_n \mathcal{F} \cap \mathbf{N} \), where \( \mathcal{H} \) is some \( \mathcal{N}_\infty \)-critical subformations of \( \mathcal{F} \);
4) every subformation of the form \( \mathcal{N}_p \mathcal{N}_q \) in \( \mathcal{F} \) has an \( \mathcal{N}_\infty \)-complement in \( \mathcal{F} \).

**References**


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