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# ANALYTICAL SOLUTION OF TWO-DIMENSIONAL CONTACT PROBLEMS OF UNSTEADY HEAT CONDUCTION IN THE PRESENCE OF MIXED BOUNDARY CONDITIONS IN THE CONTACT PLANE

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*A method for solution of systems of parabolic differential equations of heat conduction on the model of thermal contact between two bodies with different thermophysical characteristics in the presence of mixed boundary conditions in the plane of their contact has been suggested for the first time. The case of contact of two semibounded bodies has been considered. In this case, a heat source of low heat capacity acts in a circular region of finite radius on the contact surface, and beyond this region the initial temperature is maintained during the whole period of heat transfer.*

Let us assume that in a cylindrical coordinate system ($r > 0$, $-\infty < z < +\infty$) with the origin of coordinates taken in the plane of contact of the bodies under consideration it is required to solve the axisymmetric problem for the system of two differential equations of heat conduction

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\theta_1(r,z,\tau)}{\partial r}\right]+\frac{\partial^2\theta_1(r,z,\tau)}{\partial z^2}=\frac{1}{a_1}\frac{\partial\theta_1(r,z,\tau)}{\partial\tau}\,,\quad r>0\,,\quad z>0\,,\quad \tau>0\,; \tag{1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\theta_2(r,z,\tau)}{\partial r}\right]+\frac{\partial^2\theta_2(r,z,\tau)}{\partial z^2}=\frac{1}{a_2}\frac{\partial\theta_2(r,z,\tau)}{\partial\tau}\,,\quad r>0\,,\quad z<0\,,\quad \tau>0\,, \tag{2}$$

with the initial conditions

$$\theta_1(r,z,0)=\theta_2(r,z,0)=0\,,\quad r\geq 0\,,\quad -\infty<z<\infty\,, \tag{3}$$

the condition of symmetry

$$\frac{\partial\theta_1(0,z,\tau)}{\partial r}=\frac{\partial\theta_2(0,-z,\tau)}{\partial r}=0\,,\quad z\geq 0\,,\quad \tau\geq 0 \tag{4}$$

and the conditions at infinities

$$\frac{\partial\theta_1(r,\infty,\tau)}{\partial z}=\frac{\partial\theta_2(r,-\infty,\tau)}{\partial z}=\frac{\partial\theta_1(\infty,z,\tau)}{\partial r}=\frac{\partial\theta_2(\infty,-z,\tau)}{\partial r}=0\,,\quad r\geq 0\,,\quad z\geq 0\,,\quad \tau\geq 0\,; \tag{5}$$

here and in what follows the subscripts 1 and 2 refer to thermal characteristics of the first and second bodies (in our case, these bodies are half-spaces), $\theta_i(r, z, \tau) = T_i(r, z, \tau) - T_0$ are the excess temperatures and $T_i(r,$

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$z$, $\tau$) are the absolute temperatures of the corresponding bodies, $T_0$ is the initial temperature of the bodies, $a_i > 0$ are the coefficients of thermal diffusivity of the corresponding bodies, $r$ and $z$ are the cylindrical coordinates, and $\tau$ is the time variable.

In the contact plane $z = 0$, we have discontinuous mixed boundary conditions when the boundary conditions of the fourth kind [1, p. 28] with a surface circular heat source of the density $q(r, \tau)$ are specified in the region of the circle $0 < r < R$ and a constant initial temperature is maintained beyond this circle:

$$-\lambda_1 \frac{\partial\theta_1(r, 0, \tau)}{\partial z} - \lambda_2 \frac{\partial\theta_2(r, 0, \tau)}{\partial z} = q(r, \tau), \quad 0 < r < R, \quad \tau > 0; \tag{6}$$

$$\theta_1(r, 0, \tau) = \theta_2(r, 0, \tau), \quad 0 < r < R, \quad \tau > 0; \tag{7}$$

$$\theta_1(r, 0, \tau) = \theta_2(r, 0, \tau) = 0, \quad R < r < \infty, \quad \tau > 0, \tag{8}$$

where $\lambda_i > 0$ are the coefficients of thermal conductivity of the considered bodies.

We note that boundary condition (6) has the physical meaning that at any instant of time $\tau > 0$ the entire specific thermal power generated by a thin circular heat source of low heat capacity in the circle ($0 < r < R$, $z = 0$) will consist of specific heat capacities $q_1(r, \tau)$ and $q_2(r, \tau)$ which refer to the corresponding body at points of their thermal contact. Since it is known that $q_1(r, \tau) = \frac{-\lambda_1 \partial\theta_1(r, 0, \tau)}{\partial z}$ and $q_2(r, \tau) = -\lambda_2 \frac{\partial\theta_2(r, 0, \tau)}{\partial z}$, the identity is obvious:

$$q(r, \tau) = q_1(r, \tau) + q_2(r, \tau), \quad 0 < r < R, \quad \tau > 0. \tag{9}$$

The system of equations (1)–(2) with conditions (3)–(5) can be solved (see, e.g., [2 (p. 481), 3] using the corresponding integral Laplace and Hankel transforms. These solutions can be written in the form

$$\bar{\theta}_1(r, z, s) = \bar{T}_1(r, z, s) - \frac{T_0}{s} = \int_0^\infty \bar{C}_1(p, s) \exp\left(-z\sqrt{p^2 + \frac{s}{a_1}}\right) J_0(pr)\, p dp, \quad r > 0, \quad z > 0, \tag{10}$$

$$\bar{\theta}_2(r, z, s) = \bar{T}_2(r, z, s) - \frac{T_0}{s} = \int_0^\infty \bar{C}_2(p, s) \exp\left(-|z|\sqrt{p^2 + \frac{s}{a_2}}\right) J_0(pr)\, p dp, \quad r > 0, \quad z < 0, \tag{11}$$

where, for brevity, the condition $\operatorname{Re} s > 0$ for the complex parameter $s$ of the integral Laplace transform ($L$-parameter) here and in what follows is omitted and implied on default, $J_0(u)$ is the Bessel function of the real argument of the first kind, and $\bar{C}_1(p, s)$ and $\bar{C}_2(p, s)$ are unknown analytical functions which are to be determined; the Laplace transforms ($L$-transforms) of the corresponding excess temperatures are determined by the formula

$$\bar{\theta}_i(r, z, s) = L[\theta_i(r, z, \tau)] = \int_0^\infty \theta_i(r, z, \tau) \exp(-s\tau)\, d\tau, \quad r > 0, \quad |z| > 0, \quad i = 1, 2. \tag{12}$$

Using the mixed boundary conditions (6) and (8), equality (9), and the form of the solutions (10) and (11) at $z = 0$ in the corresponding ranges of variation of the cylindrical coordinate $r$, we come to the following system of paired integral equations with the $L$-parameter:

$$\lambda_i \int_0^\infty \overline{C}_i\,(p, s) \sqrt{p^2 + \frac{s}{a_i}}\, J_0\,(pr)\, p dp = \overline{q}_i\,(r, s)\,, \quad 0 < r < R\,, \tag{13}$$

$$\int_0^\infty \overline{C}_i\,(p, s)\, J_0\,(pr)\, p dp = 0\,, \quad R < r < \infty\,, \quad i = 1, 2\,. \tag{14}$$

We note that in system (13)–(14) the values of the analytical $L$-transforms $\overline{q}_1(r, s)$ and $\overline{q}_2(r, s)$ are unknown, but the corresponding sum of them is known in accordance with formula (9):

$$\overline{q}\,(r, s) = L\,[q\,(r, \tau)] = L\,[q_1\,(r, \tau) + q_2\,(r, \tau)] = L\,[q_1\,(r, \tau)] + L\,[q_2\,(r, \tau)] = \overline{q}_1\,(r, s) + \overline{q}_2\,(r, s)\,.$$

We solve the paired integral equations from system (13)–(14) with the aid of the substitution

$$\overline{C}_i\,(p, s) = \frac{1}{\sqrt{p^2 + \frac{s}{a_i}}} \int_0^R \overline{\varphi}_i\,(t, s) \sin\left( t \sqrt{p^2 + \frac{s}{a_i}} \right) dt\,, \tag{15}$$

where $\overline{\varphi}_i(t, s)$, $i = 1, 2$, are the new unknown analytical functions.

Similarly to [2, p. 184], we verify directly that for any choice of the functions $\overline{\varphi}_i(t, s)$, the substitution (15) ensures the fulfillment of the second of the paired equations of the form (14) owing to the value of the discontinuous integral (when $0 < t < R < r$)

$$\int_0^\infty \frac{p}{\sqrt{p^2 + \frac{s}{a_i}}} \sin\left( t \sqrt{p^2 + \frac{s}{a_i}} \right) J_0\,(pr)\, dp = \begin{cases} 0\,, & t < r\,; \\ \dfrac{\cos\left( \sqrt{\frac{s}{a_i}\,(t^2 - r^2)} \right)}{\sqrt{t^2 - r^2}}\,, & r < t\,. \end{cases} \tag{16}$$

Substitution of expression (15) into the first of the paired equations of the form (13) gives two integral equations with the $L$-parameter:

$$\lambda_i \int_0^R \overline{\varphi}_i\,(t, s) \int_0^\infty \sin\left( t \sqrt{p^2 + \frac{s}{a_i}} \right) J_0\,(pr)\, p dp\, dt = \overline{q}_i\,(r, s)\,, \quad 0 < r < R\,, \quad i = 1, 2\,. \tag{17}$$

Using the known relation $pJ_0(pr) = \frac{1}{r}\frac{d}{dr}[rJ_1(pr)]$ for calculation of the internal integral in (17), having integrated the left- and right-hand sides of the equations in (17) with respect to $r$ going from 0 to $r$, and having summed them up, we come to the integral equation

$$\sum_{i=1}^{2} \lambda_i \left[ \int_0^r \frac{t\bar{\varphi}_i (t, s)}{\sqrt{r^2 - t^2}} \exp\left( -\sqrt{\frac{s}{a_i} (r^2 - t^2)} \right) dt - \int_r^R \frac{t\bar{\varphi}_i (t, s)}{\sqrt{t^2 - r^2}} \sin\left( \sqrt{\frac{s}{a_i} (t^2 - r^2)} \right) dt + \right.$$

$$\left. + \int_0^R \bar{\varphi}_i (t, s) \sin\left( t \sqrt{\frac{s}{a_i}} \right) dt \right] = \int_0^r \bar{q} (\rho, s) \rho d\rho , \quad 0 < r < R . \tag{18}$$

We find the second integral equation for determination of the unknown functions $\bar{\varphi}_i(t, s)$, $i = 1, 2$, using the corresponding equality for $L$-transforms from the condition of conjugation of the excess temperatures (7). With account for representations (10), (11), and (15) and the corresponding values [2, p. 176] of the discontinuous integral (16), we have

$$\int_r^R \bar{\varphi}_1 (t, s) \frac{\cos\left( \sqrt{\frac{s}{a_1} (t^2 - r^2)} \right)}{\sqrt{t^2 - r^2}} dt = \int_r^R \bar{\varphi}_2 (t, s) \frac{\cos\left( \sqrt{\frac{s}{a_2} (t^2 - r^2)} \right)}{\sqrt{t^2 - r^2}} dt , \quad 0 < r < R . \tag{19}$$

Then, in Eq. (19) we replace $r$ by $\mu$, multiply the left- and right-hand sides by the integrating factor $\frac{2\mu}{\sqrt{\mu^2 - r^2}} \cosh\left( \sqrt{\frac{s}{a_1} (\mu^2 - r^2)} \right)$, integrate the obtained equality with respect to $\mu$ going from $r$ to $R$, and change the order of integration. As a result we have the equation

$$\int_r^R \bar{\varphi}_1 (t, s)\, dt = \int_r^R \bar{\varphi}_2 (t, s)\, B_0 (r, t, s, a_1, a_2)\, dt , \quad 0 < r < R , \tag{20}$$

where

$$B_0 (r, t, s, a_1, a_2) = \begin{cases} I_0\left( \sqrt{\frac{s\,(a_2 - a_1)}{a_1 a_2} (t^2 - r^2)} \right), & 0 < a_1 < a_2 , \\ 1 , & 0 < a_1 = a_2 , \\ J_0\left( \sqrt{\frac{s\,(a_1 - a_2)}{a_1 a_2} (t^2 - r^2)} \right), & 0 < a_2 < a_1 , \end{cases} \tag{21}$$

$I_0(u)$ is the modified Bessel function of the first kind, $0 < r < t < R$. In this case, in deriving Eq. (20) we used the value of the discontinuous integral [4, p. 477]

$$\int_0^1 \frac{\mathrm{ch}\left( \sqrt{1 - x^2} \sqrt{\frac{s}{a_1} (t^2 - r^2)} \right) \cos\left( x \sqrt{\frac{s}{a_2} (t^2 - r^2)} \right)}{\sqrt{1 - x^2}} dx = \frac{\pi}{2} B_0 (r, t, s, a_1, a_2) . \tag{22}$$

Differentiating both sides of Eq. (20) with respect to $r$, we apparently come to the formula

$$\bar{\varphi}_1 (r, s) = \bar{\varphi}_2 (r, s) + \int_r^R \bar{\varphi}_2 (t, s)\, B_1 (r, t, s, a_1, a_2)\, dt , \quad 0 < r < R , \tag{23}$$

where

$$B_1(r, t, s, a_1, a_2) = \begin{cases} r\left(\sqrt{\dfrac{s(a_2 - a_1)}{a_1 a_2 (t^2 - r^2)}}\right) I_1\left(\sqrt{\dfrac{s(a_2 - a_1)}{a_1 a_2}(t^2 - r^2)}\right), & 0 < a_1 < a_2, \\ 0, & 0 < a_1 = a_2, \\ -r\left(\sqrt{\dfrac{s(a_1 - a_2)}{a_1 a_2 (t^2 - r^2)}}\right) J_1\left(\sqrt{\dfrac{s(a_1 - a_2)}{a_1 a_2}(t^2 - r^2)}\right), & 0 < a_2 < a_1, \end{cases} \quad (24)$$

from which it follows that the unknown analytical functions $\overline{\varphi}_i(r, s)$ for the problem under consideration must be determined from (18) and (23) depending on the ratio of the coefficients of thermal diffusivity $a_1$ and $a_2$ of the considered semibounded bodies.

First, we consider the case $a_1 = a_2 = a$ where (23) quite naturally yields the equality $\overline{\varphi}_1(t, s) = \overline{\varphi}_2(t, s) = \overline{\varphi}(t, s)$ and (18) yields the equation

$$\int_0^r \frac{t\overline{\varphi}(t, s)}{\sqrt{r^2 - t^2}} \exp\left(-\sqrt{\frac{s}{a}(r^2 - t^2)}\right) dt - \int_r^R \frac{t\overline{\varphi}(t, s)}{\sqrt{t^2 - r^2}} \sin\left(\sqrt{\frac{s}{a}(t^2 - r^2)}\right) dt +$$

$$+ \int_0^R \overline{\varphi}(t, s) \sin\left(t\sqrt{\frac{s}{a}}\right) dt = \frac{1}{\lambda_1 + \lambda_2} \int_0^r \overline{q}(\rho, s)\, \rho d\rho, \quad 0 < r < R. \quad (25)$$

If in the last equation we use the integrating factor $\dfrac{2\mu}{\sqrt{r^2 - \mu^2}} \cos\left(\sqrt{\dfrac{s}{a}(r^2 - \mu^2)}\right)$, having preliminarily replaced $r$ by $\mu$ in (25), and integrate the obtained equation with respect to $\mu$ going from 0 to $r$, then, for the function $\overline{\varphi}(t, s)$ to be determined, we can write another integral equation with the $L$-parameter, which resembles the Fredholm equation of the second kind

$$\overline{\varphi}(r, s) - \frac{1}{\pi}\int_0^R \overline{\varphi}(t, s) \left[\frac{\sin\left((t - r)\sqrt{\dfrac{s}{a}}\right)}{t - r} - \frac{\sin\left((t + r)\sqrt{\dfrac{s}{a}}\right)}{t + r}\right] dt =$$

$$= \frac{1}{\pi(\lambda_1 + \lambda_2)} \int_0^r \frac{\cos\left(\sqrt{\dfrac{s}{a}(r^2 - \rho^2)}\right)}{\sqrt{r^2 - \rho^2}} \overline{q}(\rho, s)\, \rho d\rho, \quad 0 < r < R. \quad (26)$$

As an example of the method of determination of the unknown functions $\overline{\varphi}_i(r, s)$, $i = 1, 2$, in the case $a_1 \neq a_2$ we consider here the variant $a_1 < a_2$.

Let us introduce the dimensionless parameter $K_\lambda = \lambda_1/\lambda_2$ into consideration. We substitute the corresponding value of the function $\overline{\varphi}_1(t, s)$ from Eq. (23) into the integral equation (18). Then, to determine the analytical function $\overline{\varphi}_2(r, s)$ we obtain the integral equation with the $L$-parameter

$$\int_0^r \overline{\varphi}_2(t, s)\,[E_1(r, t, s) + K_\lambda^{-1} E_2(r, t, s)]\, t dt - \int_r^R \overline{\varphi}_2(t, s)\,[S_1(r, t, s) + K_\lambda^{-1} S_2(r, t, s)]\, t dt +$$

$$+\int_0^r \overline{\varphi}_2(\rho, s)\,\rho d\rho \int_0^\rho E_1(r, t, s)\,B_1(t, \rho, s, a_1, a_2)\,dt + \int_r^R \overline{\varphi}_2(\rho, s)\,\rho d\rho \int_0^r E_1(r, t, s)\times$$

$$\times B_1(t, \rho, s, a_1, a_2)\,dt - \int_r^R \overline{\varphi}_2(\rho, s)\,\rho d\rho \int_r^\rho S_1(r, t, s)\,B_1(t, \rho, s, a_1, a_2)\,dt + \int_0^R \overline{\varphi}_2(\rho, s)\,\rho d\rho \int_0^\rho S_1(0, t, s)\times$$

$$\times B_1(t, \rho, s, a_1, a_2)\,dt + \int_0^R \overline{\varphi}_2(t, s)\,[S_1(0, t, s) + K_\lambda^{-1} S_2(0, t, s)]\,tdt = \frac{1}{\lambda_1}\int_0^r \overline{q}(\mu, s)\,\mu d\mu\,, \quad 0 < r < R\,, \tag{27}$$

where

$$E_i(r, t, s) = \frac{\exp\left(-\sqrt{\frac{s}{a_i}(r^2 - t^2)}\right)}{\sqrt{r^2 - t^2}}\,, \quad S_i(r, t, s) = \frac{\sin\left(\sqrt{\frac{s}{a_i}(t^2 - r^2)}\right)}{\sqrt{t^2 - r^2}}\,, \quad i = 1, 2\,.$$

The methods of solution of the integral equations (25), (26), or (27) are suggested in [5] and are described in the monograph [2, p. 207] in detail. Here, we only note that the most preferable method for solving such integral equations is representation of the unknown function in the form of a functional series in powers $\sqrt{s}$.

Thus, for example, having determined the function $\overline{\varphi}_2(r, s)$ from (27), we find the function $\overline{\varphi}_1(r, s)$ according to formula (23). Then, using formula (15), we find the values of $\overline{C}_i(p, s)$, $i = 1, 2$, and from formulas (10) and (11) we determine the $L$-transforms of the corresponding temperature fields $\overline{\theta}_1(r, z, s)$ and $\overline{\theta}_2(r, z, s)$. Finally, using the formula of inversion of the Laplace integral, we find the inverse transforms $\theta_i(r, z, \tau) = L^{-1}[\overline{\theta}_i(r, z, s)]$, $i = 1, 2$.

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