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EINSTEIN–MAXWELL EQUATION ON FOUR-DIMENSIONAL HOMOGENEOUS SPACES

(submitted by M.A.Malakhaltsev)

Abstract. This paper presents the solutions of the Einstein–Maxwell equation on all local four-dimensional pseudo-Riemannian homogeneous spaces and the complete local classification of four-dimensional Einstein–Maxwell homogeneous spaces with an invariant pseudo-Riemannian metric of arbitrary signature.

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Introduction

We consider classification of low-dimensional homogeneous spaces as an immediate continuation of classifications obtained by Sophus Lie. Two-dimensional homogeneous spaces were classified locally by Sophus Lie [L1] and globally by G.D. Mostow [M]. (See also preprint [KTD], where the complete classification of two-dimensional homogeneous spaces, both locally and globally, is presented.) S. Lie also obtained some results in classification of three-dimensional homogeneous spaces and described all subalgebras in the Lie algebra $\mathfrak{so}(4, \mathbb{C})$ (in terms of vector fields). A detailed account of these classifications can be found in [L2]. The local classification of all three-dimensional isotropically-faithful homogeneous spaces was obtained in [KT], and the classification (local and global) of all two- and three-dimensional pseudo-Riemannian isotropically-faithful homogeneous spaces was given in [DK].

The problem of classification of four-dimensional pseudo-Riemannian homogeneous spaces is interesting from the point of view of both geometry and physics, and not only in the case of signature $(1, 3)$ (spaces of relativity theory) but also in the case of signature $(2, 2)$ (twistors).

The classification (local and global) of all four-dimensional Riemannian homogeneous spaces can be found in [B, I], and the local classification of all complex four-dimensional homogeneous spaces with an invariant symmetric bilinear form was obtained in [K1] (for a summary of results see [K]). The complete local classification of four-dimensional homogeneous spaces with an invariant pseudo-Riemannian metric of arbitrary signature was presented in [K2]. We recall this classification in Chapter I.

Let $(\overline{G}, M)$ be a homogeneous space, $G = \overline{G}_x$ the stabilizer of an arbitrary point $x \in M$, and $(\bar{\mathfrak{g}}, \mathfrak{g})$ the pair of Lie algebras corresponding to the pair $(\overline{G}, G)$ of Lie groups. Note that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ locally uniquely defines the homogeneous space $(\overline{G}, M)$. A global description of all homogeneous spaces corresponding to the given pair can be found in [M].

The Einstein–Maxwell equation on the homogeneous space $(\overline{G}, M)$ has the form:

$$r - \lambda g = m_\omega, \tag{1}$$

where g is an invariant pseudo-Riemannian metric, ω is an invariant differential 2-form on M which is closed ($d\omega = 0$) and coclosed ($d(*\omega) = 0$), λ is an arbitrary real number, r is the Ricci tensor of g, m_ω is an invariant smooth tensor field of type (0,2) on M defined by $m_\omega = \omega \circ g^{-1} \circ \omega$, where g, ω, m_ω are identified with the corresponding mappings $TM \to T^*M$.

Equation (1) is called the Einstein equation if $m_\omega \equiv 0$.

The global classification of all four-dimensional Riemannian Einstein homogeneous spaces was given in [J], and the classification of all isotropy irreducible Riemannian homogeneous spaces, which are obviously Einstein spaces, can be found in [W] (See also [Be]).

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of Lie algebras corresponding to the homogeneous space $(\overline{G}, M)$. By $\mathfrak{m}$ denote the factor space $\bar{\mathfrak{g}}/\mathfrak{g}$ endowed with the natural structure of $\mathfrak{g}$-module. Then, in terms of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$, the Einstein–Maxwell equation on the homogeneous space $(\overline{G}, M)$ will have the form:

$$R - \lambda B = M_\Omega,$$

where B is an invariant nondegenerate symmetric bilinear form on $\mathfrak{m}$, Ω is an invariant skew-symmetric bilinear form on $\mathfrak{m}$ which is closed ($d\Omega = 0$) and coclosed ($d(*\Omega) = 0$), R is the invariant symmetric bilinear form on $\mathfrak{m}$ corresponding to the Ricci tensor, λ is an arbitrary real number, M_Ω is the invariant symmetric bilinear form on $\mathfrak{m}$ corresponding to m_ω.

The solution of the Einstein–Maxwell equation on the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is any triple (B, Ω, λ) satisfying the equality $R - \lambda B = M_\Omega$.

This paper presents the solutions of the Einstein–Maxwell equation on all local four-dimensional pseudo-Riemannian homogeneous spaces. We give this results in Chapter II, 1.

A pseudo-Riemannian homogeneous space $(\overline{G}, M)$ is said to be an Einstein–Maxwell homogeneous space if the Einstein–Maxwell equation has a solution on $(\overline{G}, M)$. We can divide the four-dimensional Einstein–Maxwell homogeneous spaces into the Riemannian (with an invariant Riemannian metric), the Lorentzian (with an invariant pseudo-Riemannian metric of signature (3,1)), and the spaces of type (2,2) (with an invariant pseudo-Riemannian metric of signature (2,2)).

This paper presents the complete local classification of four-dimensional Einstein–Maxwell homogeneous spaces with an invariant pseudo-Riemannian metric of arbitrary signature. We give these classifications in Chapter II, 2.

CHAPTER I

PSEUDO-RIEMANNIAN PAIRS

1. Classification of subalgebras in the Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$ and $\mathfrak{so}(2,2)$

Preliminaries:

1. For the sake of simplicity instead of the standard notation for a subalgebra of $\mathfrak{so}(2,2)$ ($\mathfrak{so}(4)$ or $\mathfrak{so}(3,1)$) such as

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \end{pmatrix} \middle| \; x \in \mathbb{R}, \; \lambda \in [0,1] \right\}$$

we use the following notation:

$$\mathfrak{g} = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \end{pmatrix}, \quad \lambda \in [0,1].$$

Here we imply that variables denoted by Latin letters run through $\mathbb{R}$ and that parameters are denoted by small Greek letters.

2. To refer to subalgebras determined in Theorem 1 we use the following notation:

$$d.n^k,$$

where d is the dimension of the subalgebra; n is the number of the complex subalgebra $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{so}(4,\mathbb{C})$; k is the number of the real form of $\mathfrak{g}^{\mathbb{C}}$ in Theorem 1.

Theorem 1. *Any nonzero subalgebra of the Lie algebra $\mathfrak{so}(4)$ is conjugate (with respect to $\mathrm{GL}(4,\mathbb{R})$) to one and only one of the following subalgebras:*

$$\underline{\dim\ \mathfrak{g} = 1}$$

$$1.1^2 \quad \begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \\ x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \end{pmatrix}, \quad \lambda \in [0,1]$$

$$\underline{\dim\ \mathfrak{g} = 2}$$

$$2.1^3 \quad \begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \\ x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix}$$

$$\underline{\dim\ \mathfrak{g} = 3}$$

$$3.4^2 \quad \begin{pmatrix} 0 & y & -x & -z \\ -y & 0 & -z & x \\ x & z & 0 & y \\ z & -x & -y & 0 \end{pmatrix} \qquad 3.5^2 \quad \begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & z & 0 \\ -y & -z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\dim\ \mathfrak{g} = 4}$$

$$4.2^2 \quad \begin{pmatrix} 0 & y & -x & -z \\ -y & 0 & -z & -t \\ x & z & 0 & y \\ z & t & -y & 0 \end{pmatrix}$$

$$\underline{\dim\ \mathfrak{g} = 6}$$

$$6.1^2 \quad \begin{pmatrix} 0 & x & y & z \\ -x & 0 & t & u \\ -y & -t & 0 & v \\ -z & -u & -v & 0 \end{pmatrix}$$

Any nonzero subalgebra of the Lie algebra $\mathfrak{so}(3,1)$ is conjugate (with respect to $\mathrm{GL}(4,\mathbb{R})$) to one and only one of the following subalgebras:

$\underline{\dim\ \mathfrak{g} = 1}$

$$1.1^1\ (\lambda=0)\ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad 1.1^2\ (\lambda=0)\ \begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$1.1^3\ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda x \\ 0 & 0 & -x & 0 \\ 0 & \lambda x & 0 & 0 \end{pmatrix} \qquad 1.1^4\ \begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & -\lambda x & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda x \end{pmatrix}$$

$$\lambda \in]0,1] \qquad \lambda \in]0,1[$$

$$1.4^1\ \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\underline{\dim\ \mathfrak{g} = 2}$

$$2.1^2\ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & -x & 0 \\ 0 & y & 0 & 0 \end{pmatrix} \qquad 2.4^1\ \begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$2.5^2\ \begin{pmatrix} 0 & x & 0 & -y \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix}$$

$\underline{\dim\ \mathfrak{g} = 3}$

$$3.2^2\ \begin{pmatrix} x & y & 0 & -z \\ 0 & 0 & -y & -\lambda x \\ 0 & 0 & -x & 0 \\ 0 & \lambda x & z & 0 \end{pmatrix} \qquad 3.3^2\ \begin{pmatrix} 0 & y & 0 & -z \\ 0 & 0 & -y & -x \\ 0 & 0 & 0 & 0 \\ 0 & x & z & 0 \end{pmatrix}$$

$$\lambda \geqslant 0$$

$$3.5^1\ \begin{pmatrix} 2x & y & 0 & 0 \\ 2z & 0 & -2y & 0 \\ 0 & -z & -2x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad 3.5^2\ \begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & z & 0 \\ -y & -z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\underline{\dim\ \mathfrak{g} = 4}$

$$4.1^2 \quad \begin{pmatrix} x & z & 0 & -t \\ 0 & 0 & -z & -y \\ 0 & 0 & -x & 0 \\ 0 & y & t & 0 \end{pmatrix}$$

$\underline{\dim\ \mathfrak{g} = 6}$

$$6.1^3 \quad \begin{pmatrix} 0 & x & y & z \\ -x & 0 & t & u \\ -y & -t & 0 & v \\ z & u & v & 0 \end{pmatrix}$$

Any nonzero subalgebra of the Lie algebra $\mathfrak{so}(2,2)$ *is conjugate (with respect to* $\mathrm{GL}(4,\mathbb{R})$*) to one and only one of the following subalgebras:*

$\underline{\dim\ \mathfrak{g} = 1}$

$$1.1^1 \quad \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \end{pmatrix}, \lambda \in [0,1]$$

$$1.1^2 \quad \begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \\ x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \end{pmatrix}, \lambda \in [0,1]$$

$$1.2^1 \quad \begin{pmatrix} x & x & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -x & -x \end{pmatrix}$$

$$1.2^2 \quad \begin{pmatrix} 0 & -x & 0 & -x \\ x & 0 & x & 0 \\ 0 & 0 & 0 & -x \\ 0 & 0 & x & 0 \end{pmatrix}$$

$$1.3^1 \quad \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$1.4^1 \quad \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$1.1^5 \quad \begin{pmatrix} x\cos\frac{\phi}{2} & x\sin\frac{\phi}{2} & 0 & 0 \\ -x\sin\frac{\phi}{2} & x\cos\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & -x\cos\frac{\phi}{2} & -x\sin\frac{\phi}{2} \\ 0 & 0 & x\sin\frac{\phi}{2} & -x\cos\frac{\phi}{2} \end{pmatrix}, \ \phi \in]0, \frac{\pi}{2}]$$

$$1.1^6 \quad \begin{pmatrix} -x\sin\frac{\phi}{2} & x\cos\frac{\phi}{2} & 0 & 0 \\ -x\cos\frac{\phi}{2} & -x\sin\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & x\sin\frac{\phi}{2} & x\cos\frac{\phi}{2} \\ 0 & 0 & -x\cos\frac{\phi}{2} & x\sin\frac{\phi}{2} \end{pmatrix}, \ \phi \in]0, \frac{\pi}{2}[$$

$\underline{\dim\ \mathfrak{g} = 2}$

2.1[1] $\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \end{pmatrix}$ 2.1[3] $\begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \\ x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix}$

2.1[4] $\begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & -y & -x \end{pmatrix}$ 2.2[1] $\begin{pmatrix} x & y & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix}, \lambda \in [-1,1]$

2.2[2] $\begin{pmatrix} 0 & -x & y & 0 \\ x & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & x & 0 \end{pmatrix}$ 2.3[1] $\begin{pmatrix} x & y & 0 & x \\ 0 & -x & -x & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & x \end{pmatrix}$

2.4[1] $\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 2.5[1] $\begin{pmatrix} 0 & x & 0 & y \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \end{pmatrix}$

2.2[3] $$\begin{pmatrix} -x\sin\frac{\phi}{2} & x\cos\frac{\phi}{2} & y & 0 \\ -x\cos\frac{\phi}{2} & -x\sin\frac{\phi}{2} & 0 & y \\ 0 & 0 & x\sin\frac{\phi}{2} & x\cos\frac{\phi}{2} \\ 0 & 0 & -x\cos\frac{\phi}{2} & x\sin\frac{\phi}{2} \end{pmatrix}, \quad \phi \in \,]0,\pi[$$

$\underline{\dim\ \mathfrak{g} = 3}$

3.1[1] $\begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix}$ 3.1[2] $\begin{pmatrix} x & -y & z & 0 \\ y & x & 0 & z \\ 0 & 0 & -x & -y \\ 0 & 0 & y & -x \end{pmatrix}$

3.2[1] $\begin{pmatrix} x & y & 0 & z \\ 0 & \lambda x & -z & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix}, \lambda \geqslant 0$ 3.3[1] $\begin{pmatrix} 0 & y & 0 & z \\ 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -y & -x \end{pmatrix}$

3.4[1] $\begin{pmatrix} x & y & 0 & 0 \\ z & -x & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix}$ 3.5[1] $\begin{pmatrix} 2x & y & 0 & 0 \\ 2z & 0 & -2y & 0 \\ 0 & -z & -2x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\underline{\dim\ \mathfrak{g} = 4}$

4.1[1] $\begin{pmatrix} x & z & 0 & t \\ 0 & y & -t & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix}$ 4.2[1] $\begin{pmatrix} x & y & 0 & 0 \\ z & t & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & -t \end{pmatrix}$

$$4.3^1 \quad \begin{pmatrix} x & y & 0 & t \\ z & -x & -t & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix}$$

$\underline{\dim\ \mathfrak{g} = 5}$

$$5.1^1 \quad \begin{pmatrix} x & y & 0 & u \\ z & t & -u & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & -t \end{pmatrix}$$

$\underline{\dim\ \mathfrak{g} = 6}$

$$6.1^1 \quad \begin{pmatrix} x & y & 0 & u \\ z & t & -u & 0 \\ 0 & v & -x & -z \\ -v & 0 & -y & -t \end{pmatrix}$$

2. Classification of pseudo-Riemannian pairs

Preliminaries:

1. Let $\mathfrak{g}$ be one of the subalgebras of the Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$ or $\mathfrak{so}(2,2)$ ($\mathfrak{so}(p,q)$, $p+q=4$) determined in Theorem 1. We assume that the Lie algebra $\mathfrak{g}$ acts naturally on $\mathbb{R}^4$; then $(\mathfrak{g}, \mathbb{R}^4)$ is a faithful generalized module. The enumeration of the generalized modules obtained in this way coincide with that of the corresponding subalgebras of $\mathfrak{so}(p,q)$ in Theorem 1.

We say that a pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has type $n.m^k$, if the corresponding generalized module $(\mathfrak{g}, \bar{\mathfrak{g}}/\mathfrak{g})$ is isomorphic to the generalized module $n.m^k$, i.e., to the generalized module $(\mathfrak{g}, \mathbb{R}^4)$, where $\mathfrak{g}$ is the subalgebra of $\mathfrak{so}(p,q)$ supplied with the number $n.m^k$ in Theorem 1.

2. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type $n.m^k$. Then without loss of generality we can identify the Lie algebra $\mathfrak{g}$ with the subalgebra $n.m^k$ of $\mathfrak{so}(p,q)$.

Let $\{u_1, u_2, u_3, u_4\}$ be the standard basis of $\mathbb{R}^4$:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

3. We define a pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ in two ways: a) By the commutation table of the Lie algebra $\bar{\mathfrak{g}}$ only. Here by $\{e_1, \dots, e_n, u_1, u_2, u_3, u_4\}$ we denote a basis of $\bar{\mathfrak{g}}$ ($n = \dim \mathfrak{g}$). We assume that the Lie algebra $\mathfrak{g}$ is generated by $e_1, \dots, e_n$.

b) A more compact and generally accepted way, using the decomposition of the Lie algebra $\bar{\mathfrak{g}}$ in direct (semidirect) product of more simple Lie algebras.

By p, r, s, etc. we denote the parameters appearing in the process of the classification. If there are some complementary conditions on them, it is indicated just after the table. Otherwise we assume that these parameters run through $\mathbb{R}$.

4. By $\mathfrak{r}_2 = \langle p, q\rangle$ we denote the following Lie algebra: $[p, q] = p$.

By $\mathfrak{n}_3 = \langle h, p, q\rangle$ we denote the following Lie algebra:

$[,]$	h	p	q
h	0	0	0
p	0	0	h
q	0	$-h$	0 .

By $\mathfrak{n}_4 = \langle h, p_1, p_2, p_3\rangle$ we denote the following Lie algebra:

$[,]$	h	p_1	p_2	p_3
h	0	0	p_1	p_2
p_1	0	0	0	0
p_2	$-p_1$	0	0	0
p_3	$-p_2$	0	0	0 .

By $\mathfrak{n}_5 = \langle h, p_1, p_2, q_1, q_2\rangle$ we denote the following Lie algebra:

$[,]$	h	p_1	p_2	q_1	q_2
h	0	0	0	0	0
p_1	0	0	0	h	0
p_2	0	0	0	0	h
q_1	0	$-h$	0	0	0
q_2	0	0	$-h$	0	0 .

By $\tilde{\mathfrak{n}}_5 = \langle h, p_1, p_2, q_1, q_2\rangle$ we denote the following Lie algebra:

[,]	h	p_1	p_2	q_1	q_2
h	0	0	0	p_1	p_2
p_1	0	0	0	0	0
p_2	0	0	0	0	0
q_1	$-p_1$	0	0	0	h
q_2	$-p_2$	0	0	$-h$	0

.

5. We consider only the case $\mathfrak{g} \neq 0$.

Theorem 2. *Any pseudo-Riemannian pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of codimension 4 is equivalent to one and only one of the following pairs:*

1.1[1]

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \end{pmatrix} \right| x \in \mathbb{R},\ \lambda \in [0,1] \right\}$$

$\lambda = 0$

1.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} y & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y-x \end{pmatrix} \right| x, y \in \mathbb{R} \right\} \leftthreetimes \mathfrak{n}_3,\ \mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & -x \end{pmatrix} \right| x \in \mathbb{R} \right\}$$

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	u_2	0
u_2	0	0	0	0	u_2
u_3	u_3	$-u_2$	0	0	u_3
u_4	0	0	$-u_2$	$-u_3$	0

2.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} x & 0 & 0 \\ 0 & y-x & 0 \\ 0 & 0 & py \end{pmatrix} \right| x, y \in \mathbb{R},\ p \in \mathbb{R} \right\} \leftthreetimes \mathbb{R}^3,\ \mathfrak{g} = \left\{ \left. \begin{pmatrix} x & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\}$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	pu_2
u_3	u_3	0	0	0	u_3
u_4	0	0	$-pu_2$	$-u_3$	0

3.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R} \times \mathbb{R}, \;\; \mathfrak{g} = \left\{ \left[\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, x, 0 \right] \middle| \, x \in \mathbb{R} \right\}$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1+u_2	0
u_2	0	0	0	0	0
u_3	u_3	$-e_1-u_2$	0	0	0
u_4	0	0	0	0	0

4.

$$\bar{\mathfrak{g}} = (\mathfrak{g} \ltimes \mathfrak{n}_3) \times \mathbb{R}, \;\; \mathfrak{g} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & -x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	u_2	0
u_2	0	0	0	0	0
u_3	u_3	$-u_2$	0	0	0
u_4	0	0	0	0	0

5.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{r}_2, \;\; \mathfrak{g} = \mathfrak{so}(1,1)$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1	0
u_2	0	0	0	0	u_2
u_3	u_3	$-e_1$	0	0	0
u_4	0	0	$-u_2$	0	0

6.

$$\bar{\mathfrak{g}} = (\mathfrak{so}(1,1) \ltimes \mathbb{R}^2) \times \mathfrak{r}_2, \;\; \mathfrak{g} = \mathfrak{so}(1,1)$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	u_2
u_3	u_3	0	0	0	0
u_4	0	0	$-u_2$	0	0

7.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}^2, \;\; \mathfrak{g} = \mathfrak{so}(1,1)$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1	0
u_2	0	0	0	0	0
u_3	u_3	$-e_1$	0	0	0
u_4	0	0	0	0	0

$\lambda = \frac{1}{2}$

8.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2, \;\; \mathfrak{g} = \mathfrak{so}(1,1)$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	$\frac{1}{2}u_2$	$-u_3$	$-\frac{1}{2}u_4$
u_1	$-u_1$	0	0	$-2e_1$	u_2
u_2	$-\frac{1}{2}u_2$	0	0	u_4	0
u_3	u_3	$2e_1$	$-u_4$	0	0
u_4	$\frac{1}{2}u_4$	$-u_2$	0	0	0

9.

$$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \;\; \mathfrak{g} = \left\{ \left. \begin{pmatrix} \frac{x}{2} & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -\frac{x}{2} & 0 \\ 0 & 0 & 0 & -x \end{pmatrix} \right| x \in \mathbb{R} \right\}$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	$\frac{1}{2}u_2$	$-u_3$	$-\frac{1}{2}u_4$
u_1	$-u_1$	0	0	0	u_2
u_2	$-\frac{1}{2}u_2$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	$\frac{1}{2}u_4$	$-u_2$	0	0	0

$\lambda \in [0,1]$

10.

$$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \;\; \mathfrak{g} = 1.1^1$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	λu_2	$-u_3$	$-\lambda u_4$
u_1	$-u_1$	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	λu_4	0	0	0	0

1.1^2

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \\ x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \end{pmatrix} \right| x \in \mathbb{R}, \; \lambda \in [0,1] \right\}$$

$\lambda = 0$

1.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 2y & 0 & 0 \\ 0 & y & x \\ 0 & -x & y \end{pmatrix} \right| x, y \in \mathbb{R} \right\} \leftthreetimes \mathfrak{n}_3,\ \mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & -x & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\}$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	$-u_2$	u_1
u_2	0	0	0	0	$2u_2$
u_3	u_1	u_2	0	0	u_3
u_4	0	$-u_1$	$-2u_2$	$-u_3$	0

2.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} y & x & 0 \\ -x & y & 0 \\ 0 & 0 & py \end{pmatrix} \right| x, y \in \mathbb{R},\ p \in \mathbb{R} \right\} \leftthreetimes \mathbb{R}^3,\ \mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\}$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	0	u_1
u_2	0	0	0	0	pu_2
u_3	u_1	0	0	0	u_3
u_4	0	$-u_1$	$-pu_2$	$-u_3$	0

3.

$$\bar{\mathfrak{g}} = \mathfrak{su}(2) \times \mathbb{R} \times \mathbb{R},\ \mathfrak{g} = \left\{ \left. \left[\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}, x, 0 \right] \right| x \in \mathbb{R} \right\}$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	$e_1 + u_2$	0
u_2	0	0	0	0	0
u_3	u_1	$-e_1 - u_2$	0	0	0
u_4	0	0	0	0	0

4.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R} \times \mathbb{R},\ \mathfrak{g} = \left\{ \left. \left[\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}, x, 0 \right] \right| x \in \mathbb{R} \right\}$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	$-e_1+u_2$	0
u_2	0	0	0	0	0
u_3	u_1	e_1-u_2	0	0	0
u_4	0	0	0	0	0

5.

$$\bar{\mathfrak{g}} = (\mathfrak{g} \ltimes \mathfrak{n}_3) \times \mathbb{R}, \;\; \mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ 0 & x & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\}$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	u_2	0
u_2	0	0	0	0	0
u_3	u_1	$-u_2$	0	0	0
u_4	0	0	0	0	0

6.

$$\bar{\mathfrak{g}} = \mathfrak{su}(2) \times \mathfrak{r}_2, \;\; \mathfrak{g} = \mathfrak{so}(2)$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	e_1	0
u_2	0	0	0	0	u_2
u_3	u_1	$-e_1$	0	0	0
u_4	0	0	$-u_2$	0	0

7.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{r}_2, \;\; \mathfrak{g} = \mathfrak{so}(2)$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	$-e_1$	0
u_2	0	0	0	0	u_2
u_3	u_1	e_1	0	0	0
u_4	0	0	$-u_2$	0	0

8.

$\bar{\mathfrak{g}} = (\mathfrak{so}(2) \ltimes \mathbb{R}^2) \times \mathfrak{r}_2, \;\; \mathfrak{g} = \mathfrak{so}(2)$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	0	0
u_2	0	0	0	0	u_2
u_3	u_1	0	0	0	0
u_4	0	0	$-u_2$	0	0

9.

$\bar{\mathfrak{g}} = \mathfrak{su}(2) \times \mathbb{R}^2, \;\; \mathfrak{g} = \mathfrak{so}(2)$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	e_1	0
u_2	0	0	0	0	0
u_3	u_1	$-e_1$	0	0	0
u_4	0	0	0	0	0

10.

$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}^2, \;\; \mathfrak{g} = \mathfrak{so}(2)$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	$-e_1$	0
u_2	0	0	0	0	0
u_3	u_1	e_1	0	0	0
u_4	0	0	0	0	0

$\lambda = \frac{1}{2}$

11.

$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2, \;\; \mathfrak{g} = \mathfrak{so}(2)$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	$\frac{1}{2}u_4$	$-u_1$	$-\frac{1}{2}u_2$
u_1	$-u_3$	0	u_2	$-4e_1$	$-u_4$
u_2	$-\frac{1}{2}u_4$	$-u_2$	0	$-u_4$	0
u_3	u_1	$4e_1$	u_4	0	u_2
u_4	$\frac{1}{2}u_2$	u_4	0	$-u_2$	0

$\lambda \in [0,1]$

12.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \;\; \mathfrak{g} = 1.1^2$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	λu_4	$-u_1$	$-\lambda u_2$
u_1	$-u_3$	0	0	0	0
u_2	$-\lambda u_4$	0	0	0	0
u_3	u_1	0	0	0	0
u_4	λu_2	0	0	0	0

1.1^3

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda x \\ 0 & 0 & -x & 0 \\ 0 & \lambda x & 0 & 0 \end{pmatrix} \middle| \; x \in \mathbb{R}, \;\; \lambda \in]0,1] \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \;\; \mathfrak{g} = 1.1^3$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	λu_4	$-u_3$	$-\lambda u_2$
u_1	$-u_1$	0	0	0	0
u_2	$-\lambda u_4$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	λu_2	0	0	0	0

1.1^4

$$\mathfrak{g}=\left\{\left.\begin{pmatrix}0&0&-x&0\\0&-\lambda x&0&0\\x&0&0&0\\0&0&0&\lambda x\end{pmatrix}\right|\ x\in\mathbb{R},\ \lambda\in]0,1[\right\}$$

1.

$\bar{\mathfrak{g}}=\mathfrak{g}\ltimes\mathbb{R}^4,\ \ \mathfrak{g}=1.1^4$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	$-\lambda u_2$	$-u_1$	λu_4
u_1	$-u_3$	0	0	0	0
u_2	λu_2	0	0	0	0
u_3	u_1	0	0	0	0
u_4	$-\lambda u_4$	0	0	0	0

1.1^5

$$\mathfrak{g}=\left\{\left.\begin{pmatrix}x\cos\frac{\phi}{2}&x\sin\frac{\phi}{2}&0&0\\-x\sin\frac{\phi}{2}&x\cos\frac{\phi}{2}&0&0\\0&0&-x\cos\frac{\phi}{2}&-x\sin\frac{\phi}{2}\\0&0&x\sin\frac{\phi}{2}&-x\cos\frac{\phi}{2}\end{pmatrix}\right|\ x\in\mathbb{R},\ \phi\in]0,\tfrac{\pi}{2}]\right\}$$

1.

$\bar{\mathfrak{g}}=\mathfrak{g}\ltimes\mathbb{R}^4,\ \ \mathfrak{g}=1.1^5$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	$\cos\frac{\phi}{2}u_1-\sin\frac{\phi}{2}u_2$	$\sin\frac{\phi}{2}u_1+\cos\frac{\phi}{2}u_2$	$-\cos\frac{\phi}{2}u_3+\sin\frac{\phi}{2}u_4$	$-\sin\frac{\phi}{2}u_3-\cos\frac{\phi}{2}u_4$
u_1	$-\cos\frac{\phi}{2}u_1+\sin\frac{\phi}{2}u_2$	0	0	0	0
u_2	$-\sin\frac{\phi}{2}u_1-\cos\frac{\phi}{2}u_2$	0	0	0	0
u_3	$\cos\frac{\phi}{2}u_3-\sin\frac{\phi}{2}u_4$	0	0	0	0
u_4	$\sin\frac{\phi}{2}u_3+\cos\frac{\phi}{2}u_4$	0	0	0	0

1.1^6

$$\mathfrak{g}=\left\{\left.\begin{pmatrix}-x\sin\frac{\phi}{2}&x\cos\frac{\phi}{2}&0&0\\-x\cos\frac{\phi}{2}&-x\sin\frac{\phi}{2}&0&0\\0&0&x\sin\frac{\phi}{2}&x\cos\frac{\phi}{2}\\0&0&-x\cos\frac{\phi}{2}&x\sin\frac{\phi}{2}\end{pmatrix}\right|\ x\in\mathbb{R},\ \phi\in]0,\tfrac{\pi}{2}[\right\}$$

1.

$\bar{\mathfrak{g}}=\mathfrak{g}\ltimes\mathbb{R}^4,\ \ \mathfrak{g}=1.1^6$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	$-\sin\frac{\phi}{2}u_1-\cos\frac{\phi}{2}u_2$	$\cos\frac{\phi}{2}u_1-\sin\frac{\phi}{2}u_2$	$\sin\frac{\phi}{2}u_3-\cos\frac{\phi}{2}u_4$	$\cos\frac{\phi}{2}u_3+\sin\frac{\phi}{2}u_4$
u_1	$\sin\frac{\phi}{2}u_1+\cos\frac{\phi}{2}u_2$	0	0	0	0
u_2	$-\cos\frac{\phi}{2}u_1+\sin\frac{\phi}{2}u_2$	0	0	0	0
u_3	$-\sin\frac{\phi}{2}u_3+\cos\frac{\phi}{2}u_4$	0	0	0	0
u_4	$-\cos\frac{\phi}{2}u_3-\sin\frac{\phi}{2}u_4$	0	0	0	0

1.2^1

$$\mathfrak{g}=\left\{\left.\begin{pmatrix} x & x & 0 & 0\\ 0 & x & 0 & 0\\ 0 & 0 & -x & 0\\ 0 & 0 & -x & -x\end{pmatrix}\right| x\in\mathbb{R}\right\}$$

1.

$\bar{\mathfrak{g}}=\mathfrak{g}\ltimes\mathbb{R}^4,\ \ \mathfrak{g}=1.2^1$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	u_1+u_2	$-u_3-u_4$	$-u_4$
u_1	$-u_1$	0	0	0	0
u_2	$-u_1-u_2$	0	0	0	0
u_3	u_3+u_4	0	0	0	0
u_4	u_4	0	0	0	0

1.2^2

$$\mathfrak{g}=\left\{\left.\begin{pmatrix} 0 & -x & 0 & -x\\ x & 0 & x & 0\\ 0 & 0 & 0 & -x\\ 0 & 0 & x & 0\end{pmatrix}\right| x\in\mathbb{R}\right\}$$

1.

$\bar{\mathfrak{g}}=\mathfrak{g}\ltimes\mathbb{R}^4,\ \ \mathfrak{g}=1.2^2$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_2	$-u_1$	u_2+u_4	$-u_1-u_3$
u_1	$-u_2$	0	0	0	0
u_2	u_1	0	0	0	0
u_3	$-u_2-u_4$	0	0	0	0
u_4	u_1+u_3	0	0	0	0

1.3[1]

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$$

1.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \left\{ \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \middle| \; z \in \mathbb{R} \right\}$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	e_1	0	u_1	u_2
u_1	$-e_1$	0	$-\frac{1}{2}u_2$	u_3	$\frac{1}{2}u_4$
u_2	0	$\frac{1}{2}u_2$	0	$\frac{1}{2}u_4$	0
u_3	$-u_1$	$-u_3$	$-\frac{1}{2}u_4$	0	0
u_4	$-u_2$	$-\frac{1}{2}u_4$	0	0	0

2.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} y & y & \lambda x \\ 0 & 0 & -\lambda x \\ 0 & x & (\lambda+1)x \end{pmatrix} \middle| \; x, y \in \mathbb{R}, \ |\lambda| \leqslant 1 \right\} \leftthreetimes \mathbb{R}^3, \quad \mathfrak{g} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\lambda e_1+(\lambda+1)u_1+\lambda u_2$	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	$\lambda e_1-(\lambda+1)u_1-\lambda u_2$	0	0	0
u_4	$-u_2$	0	$-u_2$	0	0

$$|\lambda| \leqslant 1$$

3.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x & x \\ -x & 0 & 0 & 0 \end{pmatrix} \middle| \; x \in \mathbb{R} \right\} \leftthreetimes \left[\left\{ \begin{pmatrix} y & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; y \in \mathbb{R} \right\} \leftthreetimes \mathbb{R}^3 \right],$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	u_1	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	$-u_1$	0	0	e_1
u_4	$-u_2$	0	$-u_2$	$-e_1$	0

4.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} y & y & (1+\lambda^2)x \\ 0 & 0 & -(1+\lambda^2)x \\ 0 & x & 2\lambda x \end{pmatrix} \middle| \, x, y \in \mathbb{R},\ \lambda \geqslant 0 \right\} \ltimes \mathbb{R}^3,\ \mathfrak{g} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-(1+\lambda^2)e_1+2\lambda u_1+(1+\lambda^2)u_2$	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	$(1+\lambda^2)e_1-2\lambda u_1-(1+\lambda^2)u_2$	0	0	0
u_4	$-u_2$	0	$-u_2$	0	0

$$\lambda \geqslant 0$$

5.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} \lambda x+(1+\mu)y & y & \frac{1+\lambda^2}{\mu-1}x+\lambda y \\ -\lambda x-\mu y & 0 & -\frac{\mu+\lambda^2}{\mu-1}x-\lambda y \\ x & x & y \end{pmatrix} \middle| \, x, y \in \mathbb{R},\ \lambda \geqslant 0,\ \mu \neq 1 \right\} \ltimes$$

$\mathbb{R}^3$,

$$\mathfrak{g} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\frac{\lambda^2+\mu}{\mu-1}e_1+\frac{1+\lambda^2}{\mu-1}u_2$	$-\lambda e_1+u_1+\lambda u_2$
u_2	0	0	0	$-\lambda e_1+u_1+\lambda u_2$	$-\mu e_1+(\mu+1)u_2$
u_3	$-u_1$	$\frac{\lambda^2+\mu}{\mu-1}e_1-\frac{1+\lambda^2}{\mu-1}u_2$	$\lambda e_1-u_1-\lambda u_2$	0	0
u_4	$-u_2$	$\lambda e_1-u_1-\lambda u_2$	$\mu e_1-(\mu+1)u_2$	0	0

$$\lambda \geqslant 0,\ \mu \neq 1$$

6.

$$\bar{\mathfrak{g}} = \left\{ \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & x \\ -x & 0 & 0 & 0 \end{array} \right) \middle| x \in \mathbb{R} \right\} \leftthreetimes \left[\left\{ \left(\begin{array}{ccc} 0 & y & y \\ -y & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| y \in \mathbb{R} \right\} \leftthreetimes \mathbb{R}^3 \right],$$

$$\mathfrak{g} = \left\langle \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-u_2$	u_1
u_2	0	0	0	u_1	u_2
u_3	$-u_1$	u_2	$-u_1$	0	e_1
u_4	$-u_2$	$-u_1$	$-u_2$	$-e_1$	0

7.

$$\bar{\mathfrak{g}} = \left\{ \left(\begin{array}{ccc} \frac{x+(1+2\lambda)y}{1+\lambda} & y & \frac{y-x}{1+\lambda} \\ -\frac{x+\lambda y}{1+\lambda} & 0 & \frac{x-y}{1+\lambda} \\ \frac{x+\lambda y}{1+\lambda} & x & \frac{y+\lambda x}{1+\lambda} \end{array} \right) \middle| x, y \in \mathbb{R},\ \lambda \neq -1 \right\} \leftthreetimes \mathbb{R}^3,\ \mathfrak{g} = \left\langle \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	x	y
u_2	0	0	0	y	z
u_3	$-u_1$	$-x$	$-y$	0	0
u_4	$-u_2$	$-y$	$-z$	0	0 ,

where

$$\begin{aligned} x &= \frac{1}{1+\lambda}e_1 + \frac{\lambda}{1+\lambda}u_1 - \frac{1}{1+\lambda}u_2, \\ y &= -\frac{1}{1+\lambda}e_1 + \frac{1}{1+\lambda}u_1 + \frac{1}{1+\lambda}u_2, \\ z &= -\frac{\lambda}{1+\lambda}e_1 + \frac{\lambda}{1+\lambda}u_1 + \frac{1+2\lambda}{1+\lambda}u_2, \\ \lambda &\neq -1 \end{aligned}$$

8.

$$\bar{\mathfrak{g}} = \left\{ \left(\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & x & 0 & 0 \end{matrix} \right) \middle| \, x \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \;\; \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	u_1	u_2
u_3	$-u_1$	0	$-u_1$	0	$-u_3$
u_4	$-u_2$	0	$-u_2$	u_3	0

9.

$$\bar{\mathfrak{g}} = \left\{ \left(\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -\lambda x & 0 \\ 0 & x & 0 & x \end{matrix} \right) \middle| \, x \in \mathbb{R}, \; \lambda \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \;\; \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	λu_1	$-\lambda e_1+(\lambda+1)u_2$
u_3	$-u_1$	0	$-\lambda u_1$	0	$-\lambda u_3$
u_4	$-u_2$	0	$\lambda e_1-(\lambda+1)u_2$	λu_3	0

10.

$$\bar{\mathfrak{g}} = \left\{ \left(\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x & 0 & x \end{matrix} \right) \middle| \, x \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \;\; \mathfrak{g} = \langle p \rangle$$

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	0	0	0	e_1
u_4	$-u_2$	0	$-u_2$	$-e_1$	0

11.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -x & x & 0 \\ 0 & 0 & x & 0 \\ 0 & x & 0 & x \end{pmatrix} \middle| x \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}), \ \mathfrak{g} = \langle p \rangle$$

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	$-u_1$	e_1
u_3	$-u_1$	0	u_1	0	e_1+u_3
u_4	$-u_2$	0	$-e_1$	$-e_1-u_3$	0

12.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \mu x & 0 & 0 \\ 0 & 0 & (1-\mu)x & 0 \\ 0 & x & 0 & \lambda x \end{pmatrix} \middle| x \in \mathbb{R}, \ \lambda, \mu \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}), \ \mathfrak{g} =$$

$\langle p \rangle$

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	μu_1	$-\lambda\mu e_1+(\lambda+\mu)u_2$
u_3	$-u_1$	0	$-\mu u_1$	0	$(1-\mu)u_3$
u_4	$-u_2$	$-u_1$	$\lambda\mu e_1-(\lambda+\mu)u_2$	$(\mu-1)u_3$	0

13.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \frac{x}{2} & x & 0 \\ 0 & 0 & \frac{x}{2} & 0 \\ 0 & x & 0 & \lambda x \end{pmatrix} \middle| \, x \in \mathbb{R}, \ \lambda \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	$\frac{1}{2}u_1$	$-\frac{\lambda}{2}e_1+(\lambda+\frac{1}{2})u_2$
u_3	$-u_1$	0	$-\frac{1}{2}u_1$	0	$e_1+\frac{1}{2}u_3$
u_4	$-u_2$	$-u_1$	$\frac{\lambda}{2}e_1-(\lambda+\frac{1}{2})u_2$	$-e_1-\frac{1}{2}u_3$	0

14.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & (1-\lambda)x & x & 0 \\ 0 & 0 & \lambda x & 0 \\ 0 & x & 0 & \lambda x \end{pmatrix} \middle| \, x \in \mathbb{R}, \ \lambda \neq \tfrac{1}{2} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	$(1-\lambda)u_1$	$\lambda(\lambda-1)e_1+u_2$
u_3	$-u_1$	0	$(\lambda-1)u_1$	0	$e_1+\lambda u_3$
u_4	$-u_2$	$-u_1$	$\lambda(1-\lambda)e_1-u_2$	$-e_1-\lambda u_3$	0

$$\lambda \neq \tfrac{1}{2}$$

15.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} x & y & y \\ -y & 0 & -x \\ y & x & 2x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\} \ltimes \mathbb{R}^3, \ \ \mathfrak{g} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-e_1+2u_1$	u_2
u_2	0	0	0	u_2	$-e_1+u_1$
u_3	$-u_1$	e_1-2u_1	$-u_2$	0	0
u_4	$-u_2$	$-u_2$	e_1-u_1	0	0

16.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} x & y & y \\ y & 0 & -x \\ -y & x & 2x \end{pmatrix} \middle| x, y \in \mathbb{R} \right\} \leftthreetimes \mathbb{R}^3, \;\; \mathfrak{g} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-e_1+2u_1$	u_2
u_2	0	0	0	u_2	e_1-u_1
u_3	$-u_1$	e_1-2u_1	$-u_2$	0	0
u_4	$-u_2$	$-u_2$	$-e_1+u_1$	0	0

17.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}), \;\; \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_1
u_3	$-u_1$	0	0	0	e_1
u_4	$-u_2$	0	$-u_1$	$-e_1$	0

18.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & y & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\} \leftthreetimes \mathbb{R}^3, \;\; \mathfrak{g} = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_1
u_3	$-u_1$	0	0	0	0
u_4	$-u_2$	0	$-u_1$	0	0

19.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} x & 0 & 0 & x \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x & 0 & x \end{pmatrix} \right| x \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	u_1	$-e_1+u_1+2u_2$
u_3	$-u_1$	0	$-u_1$	0	0
u_4	$-u_2$	$-u_1$	$e_1-u_1-2u_2$	0	0

20.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & -x \\ 0 & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & x & 0 & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	u_1	u_2-u_1
u_3	$-u_1$	0	$-u_1$	0	$-u_3$
u_4	$-u_2$	0	u_1-u_2	u_3	0

21.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} x & 0 & 0 & (1-\lambda)x \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & (1-\lambda)x & 0 \\ 0 & x & 0 & x \end{pmatrix} \middle| \, x \in \mathbb{R}, \ \lambda \neq 1 \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}),$$

$\mathfrak{g} = \langle p \rangle$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	λu_1	$-\lambda e_1+(1-\lambda)u_1+(1+\lambda)u_2$
u_3	$-u_1$	0	$-\lambda u_1$	0	$(1-\lambda)u_3$
u_4	$-u_2$	$-u_1$	$\lambda e_1+(\lambda-1)u_1-(1+\lambda)u_2$	$(\lambda-1)u_3$	0

$$\lambda \neq 1$$

22.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} x & 0 & 0 & \frac{x}{2} \\ 0 & \frac{x}{2} & x & 0 \\ 0 & 0 & \frac{x}{2} & 0 \\ 0 & x & 0 & x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}), \ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	$\frac{1}{2}u_1$	$-\frac{1}{2}e_1+\frac{1}{2}u_1+\frac{3}{2}u_2$
u_3	$-u_1$	0	$-\frac{1}{2}u_1$	0	$e_1+\frac{1}{2}u_3$
u_4	$-u_2$	$-u_1$	$\frac{1}{2}e_1-\frac{1}{2}u_1-\frac{3}{2}u_2$	$-e_1-\frac{1}{2}u_3$	0

23.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} x & 0 & 0 & x \\ 0 & 0 & x & 0 \\ 0 & 0 & x & 0 \\ 0 & x & 0 & x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}), \ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	0	u_1+u_2
u_3	$-u_1$	0	0	0	e_1+u_3
u_4	$-u_2$	$-u_1$	$-u_1-u_2$	$-e_1-u_3$	0

24.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} \lambda x & y & (2\lambda-1)y \\ \frac{(2\lambda-1)y}{2(\lambda-1)} & 0 & (1-2\lambda)x \\ -\frac{y}{2(\lambda-1)} & x & 2\lambda x \end{pmatrix} \middle| \, x,y\in\mathbb{R},\ \lambda\neq 1 \right\} \ltimes \mathbb{R}^3,\ \mathfrak{g} = \left\langle \begin{pmatrix} 0\\1\\0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$(1-2\lambda)e_1+2\lambda u_1$	$(2\lambda-1)u_2$
u_2	0	0	0	λu_2	$\frac{2\lambda-1}{2(\lambda-1)}e_1-\frac{1}{2(\lambda-1)}u_1$
u_3	$-u_1$	$(2\lambda-1)e_1-2\lambda u_1$	$-\lambda u_2$	0	$(\lambda-1)u_4$
u_4	$-u_2$	$(1-2\lambda)u_2$	$\frac{1-2\lambda}{2(\lambda-1)}e_1+\frac{1}{2(\lambda-1)}u_1$	$(1-\lambda)u_4$	0

$\lambda \neq 1$

25.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} \lambda x & y & (2\lambda-1)y \\ \frac{(1-2\lambda)y}{2(\lambda-1)} & 0 & (1-2\lambda)x \\ \frac{y}{2(\lambda-1)} & x & 2\lambda x \end{pmatrix} \middle| \, x,y\in\mathbb{R},\ \lambda\neq 1 \right\} \ltimes \mathbb{R}^3,\ \mathfrak{g} = \left\langle \begin{pmatrix} 0\\1\\0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$(1-2\lambda)e_1+2\lambda u_1$	$(2\lambda-1)u_2$
u_2	0	0	0	λu_2	$\frac{1-2\lambda}{2(\lambda-1)}e_1+\frac{1}{2(\lambda-1)}u_1$
u_3	$-u_1$	$(2\lambda-1)e_1-2\lambda u_1$	$-\lambda u_2$	0	$(\lambda-1)u_4$
u_4	$-u_2$	$(1-2\lambda)u_2$	$-\frac{1-2\lambda}{2(\lambda-1)}e_1-\frac{1}{2(\lambda-1)}u_1$	$(1-\lambda)u_4$	0

$\lambda \neq 1$

26.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} \frac{x}{3} & 0 & 0 & 0 \\ x & 0 & -\frac{x}{3} & 0 \\ 0 & x & \frac{4}{3}x & 0 \\ 0 & 0 & 0 & \frac{2}{3}x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \leftthreetimes \left[\left\{ \begin{pmatrix} 0 & 0 & -\frac{y}{2} \\ 0 & 0 & \frac{3}{2}y \\ y & \frac{y}{3} & 0 \end{pmatrix} \middle| \, y \in \mathbb{R} \right\} \leftthreetimes \mathbb{R}^3 \right],$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\frac{1}{3}e_1+\frac{4}{3}u_1$	$\frac{1}{3}u_2$
u_2	0	0	0	$\frac{2}{3}u_2$	$-\frac{1}{2}e_1+\frac{3}{2}u_1$
u_3	$-u_1$	$\frac{1}{3}e_1-\frac{4}{3}u_1$	$-\frac{2}{3}u_2$	0	$e_1-\frac{1}{3}u_4$
u_4	$-u_2$	$-\frac{1}{3}u_2$	$\frac{1}{2}e_1-\frac{3}{2}u_1$	$\frac{1}{3}u_4-e_1$	0

27.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} \frac{x}{3} & 0 & 0 & 0 \\ x & 0 & -\frac{x}{3} & 0 \\ 0 & x & \frac{4}{3}x & 0 \\ 0 & 0 & 0 & \frac{2}{3}x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \leftthreetimes \left[\left\{ \begin{pmatrix} 0 & 0 & \frac{y}{2} \\ 0 & 0 & -\frac{3}{2}y \\ y & \frac{y}{3} & 0 \end{pmatrix} \middle| \, y \in \mathbb{R} \right\} \leftthreetimes \mathbb{R}^3 \right],$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\frac{1}{3}e_1+\frac{4}{3}u_1$	$\frac{1}{3}u_2$
u_2	0	0	0	$\frac{2}{3}u_2$	$\frac{1}{2}e_1-\frac{3}{2}u_1$
u_3	$-u_1$	$\frac{1}{3}e_1-\frac{4}{3}u_1$	$-\frac{2}{3}u_2$	0	$e_1-\frac{1}{3}u_4$
u_4	$-u_2$	$-\frac{1}{3}u_2$	$-\frac{1}{2}e_1+\frac{3}{2}u_1$	$\frac{1}{3}u_4-e_1$	0

28.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & y \\ x & 2x & -\frac{y}{2} \\ y & 2y & x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\} \leftthreetimes \mathbb{R}^3, \quad \mathfrak{g} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$2u_1$	$2u_2$
u_2	0	0	0	u_2	$e_1-\frac{1}{2}u_1$
u_3	$-u_1$	$-2u_1$	$-u_2$	0	u_4
u_4	$-u_2$	$-2u_2$	$\frac{1}{2}u_1-e_1$	$-u_4$	0

29.

$$\bar{\mathfrak{g}} = \left\{ \left.\begin{pmatrix} 0 & 0 & -y \\ x & 2x & \frac{y}{2} \\ y & 2y & x \end{pmatrix} \right| x, y \in \mathbb{R} \right\} \ltimes \mathbb{R}^3, \ \ \mathfrak{g} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$2u_1$	$2u_2$
u_2	0	0	0	u_2	$-e_1+\frac{1}{2}u_1$
u_3	$-u_1$	$-2u_1$	$-u_2$	0	u_4
u_4	$-u_2$	$-2u_2$	$e_1-\frac{1}{2}u_1$	$-u_4$	0

30.

$$\bar{\mathfrak{g}} = \left\{ \left.\begin{pmatrix} b_3x + c_3y & y & a_3x + b_3y \\ b_1x + c_1y & 0 & a_1x + b_1y \\ b_2x + c_2y & x & a_2x + b_2y \end{pmatrix} \right| x, y \in \mathbb{R} \right\} \ltimes \mathbb{R}^3, \ \ \mathfrak{g} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle,$$

where

$$a_1 = \frac{\lambda\mu(\lambda-1)}{\lambda+\mu-\lambda\mu}, \qquad a_2 = \frac{\lambda^2+\mu-\lambda^2\mu}{\lambda+\mu-\lambda\mu}, \qquad a_3 = \frac{\lambda(1-\lambda)}{\lambda+\mu-\lambda\mu},$$

$$b_1 = -\frac{\lambda\mu}{\lambda+\mu-\lambda\mu}, \qquad b_2 = \frac{\mu}{\lambda+\mu-\lambda\mu}, \qquad b_3 = \frac{\lambda}{\lambda+\mu-\lambda\mu},$$

$$c_1 = \frac{\lambda\mu(\mu-1)}{\lambda+\mu-\lambda\mu}, \qquad c_2 = \frac{\mu(1-\mu)}{\lambda+\mu-\lambda\mu}, \qquad c_3 = \frac{\lambda+\mu^2-\mu^2\lambda}{\lambda+\mu-\lambda\mu},$$

$$\lambda+\mu-\lambda\mu \neq 0, \quad -1 \leqslant \mu \leqslant \lambda, \quad \lambda\mu > 0$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	x	y
u_2	0	0	0	y	z
u_3	$-u_1$	$-x$	$-y$	0	0
u_4	$-u_2$	$-y$	$-z$	0	0 ,

where

$$x = \frac{\lambda\mu(\lambda-1)}{\lambda+\mu-\lambda\mu}e_1 + \frac{\lambda^2+\mu-\lambda^2\mu}{\lambda+\mu-\lambda\mu}u_1 + \frac{\lambda(1-\lambda)}{\lambda+\mu-\lambda\mu}u_2,$$

$$y = -\frac{\lambda\mu}{\lambda+\mu-\lambda\mu}e_1 + \frac{\mu}{\lambda+\mu-\lambda\mu}u_1 + \frac{\lambda}{\lambda+\mu-\lambda\mu}u_2,$$

$$z = \frac{\lambda\mu(\mu-1)}{\lambda+\mu-\lambda\mu}e_1 + \frac{\mu(1-\mu)}{\lambda+\mu-\lambda\mu}u_1 + \frac{\lambda+\mu^2-\mu^2\lambda}{\lambda+\mu-\lambda\mu}u_2,$$

$$\lambda+\mu-\lambda\mu \neq 0, \quad -1 \leqslant \mu \leqslant \lambda, \quad \lambda\mu > 0$$

31.

$\bar{\mathfrak{g}} = \tilde{\mathfrak{n}}_5$, $\mathfrak{g} = \langle h \rangle$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	0
u_3	$-u_1$	0	0	0	e_1
u_4	$-u_2$	0	0	$-e_1$	0

32.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \ \ \mathfrak{g} = 1.3^1$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	0
u_3	$-u_1$	0	0	0	0
u_4	$-u_2$	0	0	0	0

1.4^1

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{r}_2, \ \ \mathfrak{g} = \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + p \right\rangle$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	e_1
u_1	0	0	u_1	u_2	u_1
u_2	$-u_1$	$-u_1$	0	u_3	0
u_3	$-u_2$	$-u_2$	$-u_3$	0	$-u_3$
u_4	$-e_1$	$-u_1$	0	u_3	0

2.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & px & 0 & 0 \\ 0 & 0 & (p-1)x & 0 \\ 0 & 0 & 0 & (p-2)x \end{pmatrix} \right| x \in \mathbb{R}, \ p \in \mathbb{R} \right\} \ltimes \mathfrak{n}_4, \ \ \mathfrak{g} = \langle h \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	e_1
u_1	0	0	0	0	pu_1
u_2	$-u_1$	0	0	0	$(p-1)u_2$
u_3	$-u_2$	0	0	0	$(p-2)u_3$
u_4	$-e_1$	$-pu_1$	$(1-p)u_2$	$(2-p)u_3$	0

3.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 2y & 0 & 0 \\ 0 & y & x \\ 0 & x & y \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\} \ltimes \mathfrak{n}_3, \quad \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	e_1
u_1	0	0	0	0	$2u_1$
u_2	$-u_1$	0	0	e_1	u_2
u_3	$-u_2$	0	$-e_1$	0	0
u_4	$-e_1$	$-2u_1$	$-u_2$	0	0

4.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 2y & 0 & 0 \\ 0 & y & -x \\ 0 & x & y \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\} \ltimes \mathfrak{n}_3, \quad \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	e_1
u_1	0	0	0	0	$2u_1$
u_2	$-u_1$	0	0	$-e_1$	u_2
u_3	$-u_2$	0	e_1	0	0
u_4	$-e_1$	$-2u_1$	$-u_2$	0	0

5.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}, \quad \mathfrak{g} = \left\{ \left[\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}, z, 0 \right] \middle| \, z \in \mathbb{R} \right\}$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	u_1	u_2	0
u_2	$-u_1$	$-u_1$	0	u_3	0
u_3	$-u_2$	$-u_2$	$-u_3$	0	0
u_4	0	0	0	0	0

6.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & 0 & x \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \right| x \in \mathbb{R} \right\} \ltimes \mathfrak{n}_4, \quad \mathfrak{g} = \langle h \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	u_2
u_3	$-u_2$	0	0	0	u_1+u_3
u_4	0	$-u_1$	$-u_2$	$-u_1-u_3$	0

7.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & 0 & -x \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \right| x \in \mathbb{R} \right\} \ltimes \mathfrak{n}_4, \quad \mathfrak{g} = \langle h \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	u_2
u_3	$-u_2$	0	0	0	$-u_1+u_3$
u_4	0	$-u_1$	$-u_2$	u_1-u_3	0

8.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \right| x \in \mathbb{R} \right\} \ltimes \mathfrak{n}_4, \quad \mathfrak{g} = \langle h \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	u_2
u_3	$-u_2$	0	0	0	u_3
u_4	0	$-u_1$	$-u_2$	$-u_3$	0

9.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & rx & 0 \\ 0 & x & x & 0 \\ 0 & 0 & x & -px \end{pmatrix} \right| x \in \mathbb{R},\ p, r \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}),\ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	$re_1+u_2+u_4$	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2-u_4$	0	pu_4
u_4	0	0	0	$-pu_4$	0

10.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & rx & 0 \\ 0 & x & x & 0 \\ 0 & 0 & 0 & -px \end{pmatrix} \right| x \in \mathbb{R},\ p, r \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}),\ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	re_1+u_2	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2$	0	pu_4
u_4	0	0	0	$-pu_4$	0

11.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} x & 0 & 0 & -x \\ 0 & 0 & rx & 0 \\ 0 & x & x & 0 \\ 0 & 0 & x & x \end{pmatrix} \right| x \in \mathbb{R},\ r \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}),\ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	$re_1+u_2+u_4$	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2-u_4$	0	u_1-u_4
u_4	0	0	0	u_4-u_1	0

12.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} x & 0 & 0 & -x \\ 0 & 0 & rx & 0 \\ 0 & x & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \right| x \in \mathbb{R},\ r \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}),\ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	re_1+u_2	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2$	0	u_1-u_4
u_4	0	0	0	u_4-u_1	0

13.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & rx & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & -x \end{pmatrix} \right| x \in \mathbb{R},\ r \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}),\ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	re_1+u_4	0
u_3	$-u_2$	0	$-re_1-u_4$	0	u_4
u_4	0	0	0	$-u_4$	0

14.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & rx & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & -x \end{pmatrix} \right| x \in \mathbb{R},\ r \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}),\ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	re_1	0
u_3	$-u_2$	0	$-re_1$	0	u_4
u_4	0	0	0	$-u_4$	0

15.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & -x \\ 0 & 0 & x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}), \;\; \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1+u_4	0
u_3	$-u_2$	0	$-e_1-u_4$	0	u_1
u_4	0	0	0	$-u_1$	0

16.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & -x \\ 0 & 0 & -x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}), \;\; \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	$-e_1+u_4$	0
u_3	$-u_2$	0	e_1-u_4	0	u_1
u_4	0	0	0	$-u_1$	0

17.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}), \;\; \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	u_4	0
u_3	$-u_2$	0	$-u_4$	0	u_1
u_4	0	0	0	$-u_1$	0

18.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \quad \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1+u_4	0
u_3	$-u_2$	0	$-e_1-u_4$	0	0
u_4	0	0	0	0	0

19.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \quad \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	$-e_1+u_4$	0
u_3	$-u_2$	0	e_1-u_4	0	0
u_4	0	0	0	0	0

20.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \quad \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	u_4	0
u_3	$-u_2$	0	$-u_4$	0	0
u_4	0	0	0	0	0

21.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & -x \\ 0 & 0 & x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}), \ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1	0
u_3	$-u_2$	0	$-e_1$	0	u_1
u_4	0	0	0	$-u_1$	0

22.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & -x \\ 0 & 0 & -x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}), \ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	$-e_1$	0
u_3	$-u_2$	0	e_1	0	u_1
u_4	0	0	0	$-u_1$	0

23.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \leftthreetimes (\mathfrak{n}_3 \times \mathbb{R}), \ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	0
u_3	$-u_2$	0	0	0	u_1
u_4	0	0	0	$-u_1$	0

24.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1	0
u_3	$-u_2$	0	$-e_1$	0	0
u_4	0	0	0	0	0

25.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \ \ \mathfrak{g} = \langle p \rangle$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	$-e_1$	0
u_3	$-u_2$	0	e_1	0	0
u_4	0	0	0	0	0

26.

$$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \ \ \mathfrak{g} = 1.4^1$$

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	0
u_3	$-u_2$	0	0	0	0
u_4	0	0	0	0	0

2.1[1]

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\}$$

1.
$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}), \;\; \mathfrak{g} = \mathfrak{so}(1,1) \times \mathfrak{so}(1,1)$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_2$	0	0	0	e_2
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_4	0	$-e_2$	0	0

2.
$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times (\mathfrak{so}(1,1) \ltimes \mathbb{R}^2), \;\; \mathfrak{g} = \mathfrak{so}(1,1) \times \mathfrak{so}(1,1)$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_2$	0	0	0	0
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_4	0	0	0	0

3.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \;\; \mathfrak{g} = 2.1^1$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	u_3	0	0	0	0	0
u_4	0	u_4	0	0	0	0

2.1^2

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & -x & 0 \\ 0 & y & 0 & 0 \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\}$$

1.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{su}(2), \quad \mathfrak{g} = \mathfrak{so}(1,1) \times \mathfrak{so}(2)$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_4$	0	0	0	e_2
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_2	0	$-e_2$	0	0

2.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}), \quad \mathfrak{g} = \mathfrak{so}(1,1) \times \mathfrak{so}(2)$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_4$	0	0	0	$-e_2$
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_2	0	e_2	0	0

3.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times (\mathfrak{so}(2) \ltimes \mathbb{R}^2), \quad \mathfrak{g} = \mathfrak{so}(1,1) \times \mathfrak{so}(2)$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_4$	0	0	0	0
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_2	0	0	0	0

4.

$$\bar{\mathfrak{g}} = \mathfrak{su}(2) \times (\mathfrak{so}(1,1) \ltimes \mathbb{R}^2), \quad \mathfrak{g} = \mathfrak{so}(2) \times \mathfrak{so}(1,1)$$

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_4$	0	0	0	e_2
u_3	u_3	0	0	0	0	0
u_4	0	u_2	0	$-e_2$	0	0

5.

$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times (\mathfrak{so}(1,1) \ltimes \mathbb{R}^2), \;\; \mathfrak{g} = \mathfrak{so}(2) \times \mathfrak{so}(1,1)$

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_4$	0	0	0	$-e_2$
u_3	u_3	0	0	0	0	0
u_4	0	u_2	0	e_2	0	0

6.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \;\; \mathfrak{g} = 2.1^2$

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_4$	0	0	0	0
u_3	u_3	0	0	0	0	0
u_4	0	u_2	0	0	0	0

2.1^3

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \\ x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix} \right| x, y \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{su}(2) \times \mathfrak{su}(2), \;\; \mathfrak{g} = \mathfrak{so}(2) \times \mathfrak{so}(2)$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_3	0	$-u_1$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_3$	0	0	0	e_1	0
u_2	0	$-u_4$	0	0	0	e_2
u_3	u_1	0	$-e_1$	0	0	0
u_4	0	u_2	0	$-e_2$	0	0

2.

$$\bar{\mathfrak{g}} = \mathfrak{su}(2) \times \mathfrak{sl}(2,\mathbb{R}), \ \ \mathfrak{g} = \mathfrak{so}(2) \times \mathfrak{so}(2)$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_3	0	$-u_1$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_3$	0	0	0	e_1	0
u_2	0	$-u_4$	0	0	0	$-e_2$
u_3	u_1	0	$-e_1$	0	0	0
u_4	0	u_2	0	e_2	0	0

3.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}), \ \ \mathfrak{g} = \mathfrak{so}(2) \times \mathfrak{so}(2)$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_3	0	$-u_1$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_3$	0	0	0	$-e_1$	0
u_2	0	$-u_4$	0	0	0	$-e_2$
u_3	u_1	0	e_1	0	0	0
u_4	0	u_2	0	e_2	0	0

4.

$$\bar{\mathfrak{g}} = \mathfrak{su}(2) \times (\mathfrak{so}(2) \rightthreetimes \mathbb{R}^2), \ \ \mathfrak{g} = \mathfrak{so}(2) \times \mathfrak{so}(2)$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_3	0	$-u_1$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_3$	0	0	0	e_1	0
u_2	0	$-u_4$	0	0	0	0
u_3	u_1	0	$-e_1$	0	0	0
u_4	0	u_2	0	0	0	0

5.

$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times (\mathfrak{so}(2) \ltimes \mathbb{R}^2), \;\; \mathfrak{g} = \mathfrak{so}(2) \times \mathfrak{so}(2)$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_3	0	$-u_1$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_3$	0	0	0	$-e_1$	0
u_2	0	$-u_4$	0	0	0	0
u_3	u_1	0	e_1	0	0	0
u_4	0	u_2	0	0	0	0

6.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \;\; \mathfrak{g} = 2.1^3$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_3	0	$-u_1$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_3$	0	0	0	0	0
u_2	0	$-u_4$	0	0	0	0
u_3	u_1	0	0	0	0	0
u_4	0	u_2	0	0	0	0

2.1^4

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & -y & -x \end{pmatrix} \right| x, y \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}, \;\; \mathfrak{g} = \mathfrak{so}(2,\mathbb{C})_{\mathbb{R}}$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	u_2	$-u_1$	$-u_4$	u_3
u_1	$-u_1$	$-u_2$	0	0	e_1	e_2
u_2	$-u_2$	u_1	0	0	e_2	$-e_1$
u_3	u_3	u_4	$-e_1$	$-e_2$	0	0
u_4	u_4	$-u_3$	$-e_2$	e_1	0	0

2.

$\bar{\mathfrak{g}} = \mathfrak{g} \rightthreetimes \mathbb{R}^4, \;\; \mathfrak{g} = 2.1^4$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	u_2	$-u_1$	$-u_4$	u_3
u_1	$-u_1$	$-u_2$	0	0	0	0
u_2	$-u_2$	u_1	0	0	0	0
u_3	u_3	u_4	0	0	0	0
u_4	u_4	$-u_3$	0	0	0	0

2.2^1

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & y & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix} \middle| \; x, y \in \mathbb{R}, \;\lambda \in [-1, 1] \right\}$$

$\lambda = 0$

1.

$\bar{\mathfrak{g}} = \mathfrak{gl}(2, \mathbb{R}) \rightthreetimes \mathbb{R}^2, \;\; \mathfrak{g} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\}$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	$-2e_2$
u_1	$-u_1$	0	0	0	u_2	$-u_1$
u_2	0	$-u_1$	0	0	0	u_2
u_3	u_3	u_4	$-u_2$	0	0	$2u_3$
u_4	0	$2e_2$	u_1	$-u_2$	$-2u_3$	0

2.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} x-y & 0 & 0 & -y \\ 0 & py & 0 & 0 \\ 0 & 0 & (p-1)y-x & 0 \\ 0 & 0 & 0 & x+y \end{pmatrix} \middle| \, x, y \in \mathbb{R}, \ p \in \mathbb{R} \right\} \ltimes (\mathbb{R} \times \mathfrak{n}_3),$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ q \right\rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	e_2	u_4	0
u_2	0	$-u_1$	$-e_2$	0	$(p-1)u_3$	pu_4
u_3	u_3	u_4	$-u_4$	$(1-p)u_3$	0	0
u_4	0	0	0	$-pu_4$	0	0

3.

$$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes (\mathbb{R} \times (\mathbb{R}\,\mathrm{id}_{\mathbb{R}^2} \ltimes \mathbb{R}^2)), \quad \mathfrak{g} = \left\{ \begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & 0 \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_1$	0	0	u_3	u_4
u_3	u_3	u_4	0	$-u_3$	0	0
u_4	0	0	0	$-u_4$	0	0

$\lambda = 1$

4.

$$\bar{\mathfrak{g}} = \mathfrak{so}(2,1) \ltimes \mathbb{R}^3, \quad \mathfrak{g} = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	e_2	0
u_2	$-u_2$	$-u_1$	0	0	e_1	e_2
u_3	u_3	u_4	$-e_2$	$-e_1$	0	0
u_4	u_4	0	0	$-e_2$	0	0

5.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} -x & 0 & y & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\} \ltimes (\mathbb{R} \times \mathfrak{n}_3), \ \mathfrak{g} = \left\langle \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ p \right\rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	e_2	0
u_3	u_3	u_4	0	$-e_2$	0	0
u_4	u_4	0	0	0	0	0

$\lambda = -\frac{1}{2}$

6.

$$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \ \ \mathfrak{g} = \left\{ \begin{pmatrix} \frac{x}{2} & 0 & 0 & -y \\ 0 & x & y & 0 \\ 0 & 0 & -\frac{x}{2} & 0 \\ 0 & 0 & 0 & -x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$\frac{3}{2}e_2$	u_1	$-\frac{1}{2}u_2$	$-u_3$	$\frac{1}{2}u_4$
e_2	$-\frac{3}{2}e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	u_4	0	0
u_2	$\frac{1}{2}u_2$	$-u_1$	$-u_4$	0	0	0
u_3	u_3	u_4	0	0	0	0
u_4	$-\frac{1}{2}u_4$	0	0	0	0	0

$\lambda \in [-1, 1]$

7.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \ \ \mathfrak{g} = 2.2^1$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$(1-\lambda)e_2$	u_1	λu_2	$-u_3$	$-\lambda u_4$
e_2	$(\lambda-1)e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	0
u_3	u_3	u_4	0	0	0	0
u_4	λu_4	0	0	0	0	0

2.2^2

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & -x & y & 0 \\ x & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & x & 0 \end{pmatrix} \middle| \ x, y \in \mathbb{R} \right\}$$

1.

$$\bar{\mathfrak{g}} = \mathfrak{so}(2,1) \ltimes \mathbb{R}^3, \ \ \mathfrak{g} = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_2	$-u_1$	u_4	$-u_3$
e_2	0	0	0	0	u_1	u_2
u_1	$-u_2$	0	0	0	e_2	0
u_2	u_1	0	0	0	0	e_2
u_3	$-u_4$	$-u_1$	$-e_2$	0	0	$-e_1$
u_4	u_3	$-u_2$	0	$-e_2$	e_1	0

2.

$$\bar{\mathfrak{g}} = \mathfrak{so}(3) \ltimes \mathbb{R}^3, \ \ \mathfrak{g} = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_2	$-u_1$	u_4	$-u_3$
e_2	0	0	0	0	u_1	u_2
u_1	$-u_2$	0	0	0	$-e_2$	0
u_2	u_1	0	0	0	0	$-e_2$
u_3	$-u_4$	$-u_1$	e_2	0	0	e_1
u_4	u_3	$-u_2$	0	e_2	$-e_1$	0

3.

$$\bar{\mathfrak{g}} = \left\{ \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x \\ 0 & 0 & 0 & x & 0 \end{array} \right) \middle| \, x \in \mathbb{R} \right\} \leftthreetimes \tilde{\mathfrak{n}}_5, \;\; \mathfrak{g} = \left\langle \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right), \; h \right\rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_2	$-u_1$	u_4	$-u_3$
e_2	0	0	0	0	u_1	u_2
u_1	$-u_2$	0	0	0	0	0
u_2	u_1	0	0	0	0	0
u_3	$-u_4$	$-u_1$	0	0	0	e_2
u_4	u_3	$-u_2$	0	0	$-e_2$	0

4.

$$\bar{\mathfrak{g}} = \mathfrak{g} \leftthreetimes \mathbb{R}^4, \;\; \mathfrak{g} = 2.2^2$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_2	$-u_1$	u_4	$-u_3$
e_2	0	0	0	0	u_1	u_2
u_1	$-u_2$	0	0	0	0	0
u_2	u_1	0	0	0	0	0
u_3	$-u_4$	$-u_1$	0	0	0	0
u_4	u_3	$-u_2$	0	0	0	0

2.2^3

$$\mathfrak{g} = \left\{ \begin{pmatrix} -x\sin\frac{\phi}{2} & x\cos\frac{\phi}{2} & y & 0 \\ -x\cos\frac{\phi}{2} & -x\sin\frac{\phi}{2} & 0 & y \\ 0 & 0 & x\sin\frac{\phi}{2} & x\cos\frac{\phi}{2} \\ 0 & 0 & -x\cos\frac{\phi}{2} & x\sin\frac{\phi}{2} \end{pmatrix} \middle| \; x,y\in\mathbb{R},\ \phi\in\,]0,\pi[\right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \ \ \mathfrak{g} = 2.2^3$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$-2\sin\frac{\phi}{2}e_2$	A	B	C	D
e_2	$2\sin\frac{\phi}{2}e_2$	0	0	0	u_1	u_2
u_1	$-A$	0	0	0	0	0
u_2	$-B$	0	0	0	0	0
u_3	$-C$	$-u_1$	0	0	0	0
u_4	$-D$	$-u_2$	0	0	0	0 ,

where

$$\begin{cases} A = -\sin\frac{\phi}{2}u_1 - \cos\frac{\phi}{2}u_2, \\ B = \cos\frac{\phi}{2}u_1 - \sin\frac{\phi}{2}u_2, \\ C = \sin\frac{\phi}{2}u_3 - \cos\frac{\phi}{2}u_4, \\ D = \cos\frac{\phi}{2}u_3 + \sin\frac{\phi}{2}u_4 \end{cases}$$

2.3^1

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & y & 0 & x \\ 0 & -x & -x & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & x \end{pmatrix} \middle| \; x,y\in\mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \ \ \mathfrak{g} = 2.3^1$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	u_1	$-u_2$	$-u_2-u_3$	u_1+u_4
e_2	$-2e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0
u_3	u_2+u_3	u_4	0	0	0	0
u_4	$-u_1-u_4$	0	0	0	0	0

2.4[1]

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| x, y \in \mathbb{R} \right\}$$

1.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{r}_2 \times \mathbb{R}, \ \ \mathfrak{g} = \left\langle \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} - q, \ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + p \right\rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	u_1	u_2	0
u_2	0	$-u_1$	$-u_1$	0	u_3	0
u_3	u_3	$-u_2$	$-u_2$	$-u_3$	0	0
u_4	0	0	0	0	0	0

2.

$$\bar{\mathfrak{g}} = (\mathfrak{g} \oplus \mathbb{R}\,\mathrm{id}_{\mathbb{R}^3}) \leftthreetimes \mathbb{R}^3, \ \ \mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 \\ 0 & 0 & y \\ 0 & 0 & -x \end{pmatrix} \right| x, y \in \mathbb{R} \right\}$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	0	0	u_1
u_2	0	$-u_1$	0	0	0	u_2
u_3	u_3	$-u_2$	0	0	0	u_3
u_4	0	0	$-u_1$	$-u_2$	$-u_3$	0

3.

$$\bar{\mathfrak{g}} = \mathfrak{g} \leftthreetimes \mathbb{R}^4, \ \ \mathfrak{g} = 2.4^1$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_1$	0	0	0	0
u_3	u_3	$-u_2$	0	0	0	0
u_4	0	0	0	0	0	0

2.5[1]

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & x & 0 & y \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\}$$

1.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & y \\ 0 & z & -x \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\} \ltimes \mathfrak{n}_3, \;\; \mathfrak{g} = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + h, \;\; p \right\rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	$-2e_1$
e_2	0	0	0	$-2e_2$	$-u_2$	u_1
u_1	0	0	0	$2e_2 - u_1$	$u_2 + u_4$	$2e_1 - u_1$
u_2	$-u_1$	$2e_2$	$u_1 - 2e_2$	0	$-2u_3$	$u_2 - u_4$
u_3	u_4	u_2	$-u_2 - u_4$	$2u_3$	0	$2u_3$
u_4	$2e_1$	$-u_1$	$u_1 - 2e_1$	$u_4 - u_2$	$-2u_3$	0

2.

$$\bar{\mathfrak{g}} = (\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2) \times \mathbb{R}, \;\; \mathfrak{g} = \left\{ \left[\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -t \\ 0 \end{pmatrix}, t \right] \middle| \; y, t \in \mathbb{R} \right\}$$

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	$-2e_2$	$-u_2$	u_1
u_1	0	0	0	$-u_1$	u_4	0
u_2	$-u_1$	$2e_2$	u_1	0	$-2u_3$	$-u_4$
u_3	u_4	u_2	$-u_4$	$2u_3$	0	0
u_4	0	$-u_1$	0	u_4	0	0

3.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & gx \\ 0 & 0 & -hx & gx & -kx \\ 0 & 0 & -x & x & 0 \\ 0 & -x & 0 & 0 & (1+h)x \end{pmatrix} \right| \begin{matrix} x \in \mathbb{R},\ g, k \in \mathbb{R}, \\ h \geqslant 0 \ (\text{if } k \neq 0), \\ h \in \mathbb{R} \ (\text{if } k = 0) \end{matrix} \right\} \leftthreetimes \mathfrak{n}_5,$$

$$\mathfrak{g} = \langle p_1,\ p_2 \rangle$$

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	u_1	0
u_2	$-u_1$	0	0	0	$e_1+ge_2+(1-h)u_2$	hu_1
u_3	u_4	u_2	$-u_1$	$-e_1-ge_2+(h-1)u_2$	0	$-(g+h)e_1+ke_2-(1+h)u_4$
u_4	0	$-u_1$	0	$-hu_1$	$(g+h)e_1-ke_2+(1+h)u_4$	0

$$h \geqslant 0 \ (\text{if } k \neq 0),\ \ h \in \mathbb{R} \ (\text{if } k = 0)$$

4.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & gx \\ 0 & 0 & -hx & gx & 0 \\ 0 & 0 & -x & x & 0 \\ 0 & -x & 0 & 0 & (1+h)x \end{pmatrix} \right| x \in \mathbb{R},\ g \in \mathbb{R},\ h \geqslant 0 \right\} \leftthreetimes \mathfrak{n}_5,$$

$$\mathfrak{g} = \langle p_1,\ p_2 \rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	u_1	0
u_2	$-u_1$	0	0	0	$ge_2+(1-h)u_2$	hu_1
u_3	u_4	u_2	$-u_1$	$-ge_2+(h-1)u_2$	0	$-(g+h)e_1-(1+h)u_4$
u_4	0	$-u_1$	0	$-hu_1$	$(g+h)e_1+(1+h)u_4$	0

$h \geqslant 0$

5.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & gx \\ 0 & 0 & -x & gx & -kx \\ 0 & 0 & -x & 0 & 0 \\ 0 & -x & 0 & 0 & x \end{pmatrix} \middle|\; x \in \mathbb{R},\ g, k \in \mathbb{R} \right\} \ltimes \mathfrak{n}_5,\ \ \mathfrak{g} = \langle p_1,\ p_2 \rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	$e_1+ge_2-u_2$	u_1
u_3	u_4	u_2	0	$-e_1-ge_2+u_2$	0	$-ge_1+ke_2-u_4$
u_4	0	$-u_1$	0	$-u_1$	$ge_1-ke_2+u_4$	0

6.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & gx \\ 0 & 0 & -x & gx & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & -x & 0 & 0 & x \end{pmatrix} \middle|\; x \in \mathbb{R},\ g \in \mathbb{R} \right\} \ltimes \mathfrak{n}_5,\ \ \mathfrak{g} = \langle p_1,\ p_2 \rangle$$

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	ge_2-u_2	u_1
u_3	u_4	u_2	0	$-ge_2+u_2$	0	$-ge_1-u_4$
u_4	0	$-u_1$	0	$-u_1$	ge_1+u_4	0

7.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & -kx \\ 0 & 0 & -x & 0 & 0 \\ 0 & -x & 0 & 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{R}, \ k \in \mathbb{R} \right\} \leftthreetimes \mathfrak{n}_5, \ \ \mathfrak{g} = \langle p_1, \ p_2 \rangle$$

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1+e_2	0
u_3	u_4	u_2	0	$-e_1-e_2$	0	$-e_1+ke_2$
u_4	0	$-u_1$	0	0	e_1-ke_2	0

8.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & -x \\ 0 & 0 & 0 & -x & -kx \\ 0 & 0 & -x & 0 & 0 \\ 0 & -x & 0 & 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{R}, \ k \in \mathbb{R} \right\} \leftthreetimes \mathfrak{n}_5, \ \ \mathfrak{g} = \langle p_1, \ p_2 \rangle$$

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1-e_2	0
u_3	u_4	u_2	0	$-e_1+e_2$	0	e_1+ke_2
u_4	0	$-u_1$	0	0	$-e_1-ke_2$	0

9.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & x & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & -x & 0 & 0 & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\} \ltimes \mathfrak{n}_5, \quad \mathfrak{g} = \langle p_1, \ p_2 \rangle$$

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_2	0
u_3	u_4	u_2	0	$-e_2$	0	$-e_1$
u_4	0	$-u_1$	0	0	e_1	0

10.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x \\ 0 & 0 & 0 & -x & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & -x & 0 & 0 & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\} \ltimes \mathfrak{n}_5, \quad \mathfrak{g} = \langle p_1, \ p_2 \rangle$$

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	$-e_2$	0
u_3	u_4	u_2	0	e_2	0	e_1
u_4	0	$-u_1$	0	0	$-e_1$	0

11.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & -x \\ 0 & 0 & -x & 0 & 0 \\ 0 & -x & 0 & 0 & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\} \ltimes \mathfrak{n}_5, \quad \mathfrak{g} = \langle p_1, \ p_2 \rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1	0
u_3	u_4	u_2	0	$-e_1$	0	e_2
u_4	0	$-u_1$	0	0	$-e_2$	0

12.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & x \\ 0 & 0 & -x & 0 & 0 \\ 0 & -x & 0 & 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \leftthreetimes \mathfrak{n}_5, \quad \mathfrak{g} = \langle p_1, \ p_2 \rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1	0
u_3	u_4	u_2	0	$-e_1$	0	$-e_2$
u_4	0	$-u_1$	0	0	e_2	0

13.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & -x & 0 & 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \leftthreetimes \mathfrak{n}_5, \quad \mathfrak{g} = \langle p_1, \ p_2 \rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1	0
u_3	u_4	u_2	0	$-e_1$	0	0
u_4	0	$-u_1$	0	0	0	0

14.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \ \ \mathfrak{g} = 2.5^1$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	0	0
u_3	u_4	u_2	0	0	0	0
u_4	0	$-u_1$	0	0	0	0

2.5^2

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & x & 0 & -y \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix} \right| x, y \in \mathbb{R} \right\}$$

1.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & y \\ 0 & z & -x \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\} \ltimes \mathfrak{n}_3, \ \ \mathfrak{g} = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - h, \ p \right\rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	$-e_1 + u_1$	$-u_2$	e_2
e_2	0	0	0	$-e_2$	u_4	$-e_1 - u_1$
u_1	0	0	0	$e_1 - u_1$	u_2	$-e_2$
u_2	$e_1 - u_1$	e_2	$-e_1 + u_1$	0	$-2u_3$	$-u_4$
u_3	u_2	$-u_4$	$-u_2$	$2u_3$	0	0
u_4	$-e_2$	$e_1 + u_1$	e_2	u_4	0	0

2.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 0 & rx & (p+s)x & 0 \\ 0 & 0 & x & 0 & x \\ 0 & -x & 2rx & x & 0 \\ 0 & 2rx & (s-p-4r^2)x & -rx & 0 \end{pmatrix} \right| \begin{matrix} x \in \mathbb{R}, \ p \in \mathbb{R}, \\ r, s \geqslant 0 \end{matrix} \right\} \ltimes \mathfrak{n}_5,$$

$\mathfrak{g} = \langle p_1, \ q_2 \rangle$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	u_1	0
u_2	$-u_1$	0	0	0	A	$2ru_1$
u_3	u_2	$-u_4$	$-u_1$	$-A$	0	B
u_4	0	u_1	0	$-2ru_1$	$-B$	0

,

where

$$\begin{cases} A = (p+s)e_1 + re_2 + u_2 - 2ru_4, \\ B = -re_1 + (p-s)e_2 - 2ru_2 - u_4, \\ r \geqslant 0,\ s \geqslant 0 \end{cases}$$

3.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(r+s)x & 0 \\ 0 & 0 & 0 & 0 & x \\ 0 & -x & x & 0 & 0 \\ 0 & x & (r-s-1)x & 0 & 0 \end{pmatrix} \middle|\, x \in \mathbb{R},\ r \in \mathbb{R},\ s \geqslant 0 \right\} \leftthreetimes$$

$\mathfrak{n}_5$,

$$\mathfrak{g} = \langle p_1,\ q_2 \rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	$-(r+s)e_1 - u_4$	u_1
u_3	u_2	$-u_4$	0	$(r+s)e_1 + u_4$	0	$(s-r)e_2 - u_2$
u_4	0	u_1	0	$-u_1$	$(r-s)e_2 + u_2$	0

$s \geqslant 0$

4.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1+s)x & 0 \\ 0 & 0 & 0 & 0 & x \\ 0 & -x & 0 & 0 & 0 \\ 0 & 0 & (s-1)x & 0 & 0 \end{pmatrix} \middle|\, x \in \mathbb{R},\ s \geqslant 0 \right\} \leftthreetimes \mathfrak{n}_5,\quad \mathfrak{g} =$$

$\langle p_1,\ q_2 \rangle$

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	$(1+s)e_1$	0
u_3	u_2	$-u_4$	0	$-(1+s)e_1$	0	$(1-s)e_2$
u_4	0	u_1	0	0	$(s-1)e_2$	0

$s \geqslant 0$

5.

$$\bar{\mathfrak{g}} = \left\{ \left.\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1+s)x & 0 \\ 0 & 0 & 0 & 0 & x \\ 0 & -x & 0 & 0 & 0 \\ 0 & 0 & (1-s)x & 0 & 0 \end{pmatrix} \right| x \in \mathbb{R},\ s \geqslant 0 \right\} \ltimes \mathfrak{n}_5, \quad \mathfrak{g} = \langle p_1,\ q_2 \rangle$$

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	$-(1+s)e_1$	0
u_3	u_2	$-u_4$	0	$(1+s)e_1$	0	$(s-1)e_2$
u_4	0	u_1	0	0	$(1-s)e_2$	0

$s \geqslant 0$

6.

$$\bar{\mathfrak{g}} = \left\{ \left.\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & 0 & x \\ 0 & -x & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\} \ltimes \mathfrak{n}_5, \quad \mathfrak{g} = \langle p_1,\ q_2 \rangle$$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_2	0
u_3	u_2	$-u_4$	0	$-e_2$	0	e_1
u_4	0	u_1	0	0	$-e_1$	0

7.

$\bar{\mathfrak{g}} = \mathfrak{g} \rightthreetimes \mathbb{R}^4, \;\; \mathfrak{g} = 2.5^2$

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	0	0
u_3	u_2	$-u_4$	0	0	0	0
u_4	0	u_1	0	0	0	0

3.1^1

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{g} \rightthreetimes \mathbb{R}^4, \;\; \mathfrak{g} = 3.1^1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	0	e_3	u_1	0	$-u_3$	0
e_2	0	0	$-e_3$	0	u_2	0	$-u_4$
e_3	$-e_3$	e_3	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0	0
u_2	0	$-u_2$	$-u_1$	0	0	0	0
u_3	u_3	0	u_4	0	0	0	0
u_4	0	u_4	0	0	0	0	0

3.1²

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & -y & z & 0 \\ y & x & 0 & z \\ 0 & 0 & -x & -y \\ 0 & 0 & y & -x \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \quad \mathfrak{g} = 3.1^2$

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	u_2	$-u_1$	u_4	$-u_3$
e_3	$-2e_3$	0	0	0	0	u_1	u_2
u_1	$-u_1$	$-u_2$	0	0	0	0	0
u_2	$-u_2$	u_1	0	0	0	0	0
u_3	u_3	$-u_4$	$-u_1$	0	0	0	0
u_4	u_4	u_3	$-u_2$	0	0	0	0

3.2¹

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 & z \\ 0 & \lambda x & -z & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix} \right| x, y, z \in \mathbb{R}, \ \lambda \geqslant 0 \right\}$$

$\lambda = 0$

1.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} t & 0 & 0 \\ 0 & t+x & y \\ 0 & z & -x \end{pmatrix} \right| x, y, z, t \in \mathbb{R} \right\} \ltimes \mathfrak{n}_3,$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + h, \ p \right\rangle$$

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_2	e_3	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	0	u_1	$-u_4$	$-2e_2$
e_3	$-e_3$	0	0	0	$-2e_3$	$-u_2$	u_1
u_1	$-u_1$	0	0	0	$2e_3-u_1$	u_2+u_4	$2e_2-u_1$
u_2	0	$-u_1$	$2e_3$	u_1-2e_3	0	$-2u_3$	u_2-u_4
u_3	u_3	u_4	u_2	$-u_2-u_4$	$2u_3$	0	$2u_3$
u_4	0	$2e_2$	$-u_1$	u_1-2e_2	u_4-u_2	$-2u_3$	0

2.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t+x & y \\ 0 & z & -x \end{pmatrix} \middle| x, y, z, t \in \mathbb{R} \right\} \leftthreetimes \mathbb{R}^3,$$

$$\mathfrak{g} = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t & y \\ 0 & 0 & 0 \end{pmatrix} \leftthreetimes \begin{pmatrix} u \\ -u \\ 0 \end{pmatrix} \middle| y, t, u \in \mathbb{R} \right\}$$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_2	e_3	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	0	u_1	$-u_4$	$-2e_2$
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	u_2	$-u_1$
u_2	0	$-u_1$	0	0	0	0	u_2
u_3	u_3	u_4	u_2	$-u_2$	0	0	$2u_3$
u_4	0	$2e_2$	$-u_1$	u_1	$-u_2$	$-2u_3$	0

$\lambda = 1$

3.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y & 0 \\ 0 & 0 & 2x & 0 & 0 \\ 0 & 0 & y & x & 0 \\ 0 & y & 0 & 0 & -x \end{pmatrix} \middle| x, y \in \mathbb{R} \right\} \leftthreetimes \mathfrak{n}_5,$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \ p_1, \ p_2 \right\rangle$$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	0	u_1	$-u_4$	0
e_3	$-2e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	e_2	0
u_3	u_3	u_4	u_2	0	$-e_2$	0	0
u_4	u_4	0	$-u_1$	0	0	0	0

$\lambda \geqslant 0$

4.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \ \ \mathfrak{g} = 3.2^1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$(1-\lambda)e_2$	$(1+\lambda)e_3$	u_1	λu_2	$-u_3$	$-\lambda u_4$
e_2	$(\lambda-1)e_2$	0	0	0	u_1	$-u_4$	0
e_3	$-(1+\lambda)e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	0	0
u_3	u_3	u_4	u_2	0	0	0	0
u_4	λu_4	0	$-u_1$	0	0	0	0

3.2^2

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 & -z \\ 0 & 0 & -y & -\lambda x \\ 0 & 0 & -x & 0 \\ 0 & \lambda x & z & 0 \end{pmatrix} \right| x, y, z \in \mathbb{R}, \ \lambda \geqslant 0 \right\}$$

$\lambda = 0$

1.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} t & 0 & 0 \\ 0 & t+x & y \\ 0 & z & -x \end{pmatrix} \right| x, y, z, t \in \mathbb{R} \right\} \ltimes \mathfrak{n}_3,$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - h, \ p \right\rangle$$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_2	e_3	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	0	$-e_2+u_1$	$-u_2$	e_3
e_3	$-e_3$	0	0	0	$-e_3$	u_4	$-e_2-u_1$
u_1	$-u_1$	0	0	0	e_2-u_1	u_2	$-e_3$
u_2	0	e_2-u_1	e_3	$-e_2+u_1$	0	$-2u_3$	$-u_4$
u_3	u_3	u_2	$-u_4$	$-u_2$	$2u_3$	0	0
u_4	0	$-e_3$	e_2+u_1	e_3	u_4	0	0

$\lambda \geqslant 0$

2.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \;\; \mathfrak{g} = 3.2^2$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$e_2 - \lambda e_3$	$e_3 + \lambda e_2$	u_1	λu_4	$-u_3$	$-\lambda u_2$
e_2	$\lambda e_3 - e_2$	0	0	0	u_1	$-u_2$	0
e_3	$-e_3 - \lambda e_2$	0	0	0	0	u_4	$-u_1$
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-\lambda u_4$	$-u_1$	0	0	0	0	0
u_3	u_3	u_2	$-u_4$	0	0	0	0
u_4	λu_2	0	u_1	0	0	0	0

3.3^1

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & y & 0 & z \\ 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -y & -x \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}$$

1.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} y & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & py \\ 0 & 0 & x & py & 0 \\ 0 & 0 & -y & x+y & 0 \\ 0 & -y & 0 & 0 & y-x \end{pmatrix} \right| x, y \in \mathbb{R}, \; p \in \mathbb{R} \right\} \ltimes \mathfrak{n}_5,$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \; p_1, \; p_2 \right\rangle$$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	u_1	0
u_2	$-u_2$	$-u_1$	0	0	0	$pe_3 + u_2$	0
u_3	0	u_4	u_2	$-u_1$	$-pe_3 - u_2$	0	$-pe_2 - u_4$
u_4	u_4	0	$-u_1$	0	0	$pe_2 + u_4$	0

2.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & y \\ 0 & 0 & x & y & 0 \\ 0 & 0 & -y & x & 0 \\ 0 & -y & 0 & 0 & -x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\} \ltimes \mathfrak{n}_5,$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \ p_1, \ p_2 \right\rangle$$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	e_3	0
u_3	0	u_4	u_2	0	$-e_3$	0	$-e_2$
u_4	u_4	0	$-u_1$	0	0	e_2	0

3.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & -y \\ 0 & 0 & x & -y & 0 \\ 0 & 0 & -y & x & 0 \\ 0 & -y & 0 & 0 & -x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\} \ltimes \mathfrak{n}_5,$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \ p_1, \ p_2 \right\rangle$$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	$-e_3$	0
u_3	0	u_4	u_2	0	e_3	0	e_2
u_4	u_4	0	$-u_1$	0	0	$-e_2$	0

4.

$\bar{\mathfrak{g}} = \mathfrak{g} \leftthreetimes \mathbb{R}^4, \;\; \mathfrak{g} = 3.3^1$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	0	0
u_3	0	u_4	u_2	0	0	0	0
u_4	u_4	0	$-u_1$	0	0	0	0

3.3^2

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & y & 0 & -z \\ 0 & 0 & -y & -x \\ 0 & 0 & 0 & 0 \\ 0 & x & z & 0 \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}$$

1.

$$\bar{\mathfrak{g}} = \left\{ \left. \begin{pmatrix} y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & py & x \\ 0 & 0 & y & x & y \\ 0 & -y & -x & y & 0 \\ 0 & -x & -py & 0 & 0 \end{pmatrix} \right| x, y \in \mathbb{R}, \; p \in \mathbb{R} \right\} \leftthreetimes \mathfrak{n}_5,$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \; p_1, \; q_2 \right\rangle$$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	0	u_4	0	$-u_2$
e_2	e_3	0	0	0	u_1	$-u_2$	0
e_3	$-e_2$	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	u_1	0
u_2	$-u_4$	$-u_1$	0	0	0	pe_2+u_2	0
u_3	0	u_2	$-u_4$	$-u_1$	$-pe_2-u_2$	0	pe_3-u_4
u_4	u_2	0	u_1	0	0	$-pe_3+u_4$	0

2.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & x \\ 0 & 0 & 0 & x & y \\ 0 & -y & -x & 0 & 0 \\ 0 & -x & -y & 0 & 0 \end{pmatrix} \middle|\, x,y \in \mathbb{R} \right\} \ltimes \mathfrak{n}_5,$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},\ p_1,\ q_2 \right\rangle$$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	0	u_4	0	$-u_2$
e_2	e_3	0	0	0	u_1	$-u_2$	0
e_3	$-e_2$	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0	0
u_2	$-u_4$	$-u_1$	0	0	0	e_2	0
u_3	0	u_2	$-u_4$	0	$-e_2$	0	e_3
u_4	u_2	0	u_1	0	0	$-e_3$	0

3.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y & x \\ 0 & 0 & 0 & x & y \\ 0 & -y & -x & 0 & 0 \\ 0 & -x & y & 0 & 0 \end{pmatrix} \middle|\, x,y \in \mathbb{R} \right\} \ltimes \mathfrak{n}_5,$$

$$\mathfrak{g} = \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},\ p_1,\ q_2 \right\rangle$$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	0	u_4	0	$-u_2$
e_2	e_3	0	0	0	u_1	$-u_2$	0
e_3	$-e_2$	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0	0
u_2	$-u_4$	$-u_1$	0	0	0	$-e_2$	0
u_3	0	u_2	$-u_4$	0	e_2	0	$-e_3$
u_4	u_2	0	u_1	0	0	e_3	0

4.

$\bar{\mathfrak{g}} = \mathfrak{g} \leftthreetimes \mathbb{R}^4, \;\; \mathfrak{g} = 3.3^2$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	0	u_4	0	$-u_2$
e_2	e_3	0	0	0	u_1	$-u_2$	0
e_3	$-e_2$	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0	0
u_2	$-u_4$	$-u_1$	0	0	0	0	0
u_3	0	u_2	$-u_4$	0	0	0	0
u_4	u_2	0	u_1	0	0	0	0

3.4^1

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & y & 0 & 0 \\ z & -x & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \ltimes (V \oplus V^*)$, where $V = \mathbb{R}^2$ is natural $\mathfrak{sl}(2,\mathbb{R})$-module,
$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	0	$-u_2$	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0
u_3	u_3	u_4	0	0	0	0	0
u_4	$-u_4$	0	u_3	0	0	0	0

3.4^2

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & y & -x & -z \\ -y & 0 & -z & x \\ x & z & 0 & y \\ z & -x & -y & 0 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4$, $\mathfrak{g} = 3.4^2 \cong \mathfrak{su}(2)$, $\mathfrak{g}$ acts irreducibly on $\mathbb{R}^4$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_3$	$-2e_2$	u_3	$-u_4$	$-u_1$	u_2
e_2	$-2e_3$	0	$2e_1$	$-u_2$	u_1	$-u_4$	u_3
e_3	$2e_2$	$-2e_1$	0	u_4	u_3	$-u_2$	$-u_1$
u_1	$-u_3$	u_2	$-u_4$	0	0	0	0
u_2	u_4	$-u_1$	$-u_3$	0	0	0	0
u_3	u_1	u_4	u_2	0	0	0	0
u_4	$-u_2$	$-u_3$	u_1	0	0	0	0

3.5^1

$$\mathfrak{g} = \left\{ \begin{pmatrix} 2x & y & 0 & 0 \\ 2z & 0 & -2y & 0 \\ 0 & -z & -2x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = (\mathfrak{so}(2,1) \oplus \mathbb{R}\,\mathrm{id}_{\mathbb{R}^3}) \ltimes \mathbb{R}^3$, $\mathfrak{g} = \mathfrak{so}(2,1)$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	0	0	u_1
u_2	0	$-u_1$	u_3	0	0	0	u_2
u_3	$2u_3$	$2u_2$	0	0	0	0	u_3
u_4	0	0	0	$-u_1$	$-u_2$	$-u_3$	0

2.

$\bar{\mathfrak{g}} = \mathfrak{so}(2,2) \times \mathbb{R}, \;\; \mathfrak{g} = \mathfrak{so}(2,1)$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	e_2	e_1	0
u_2	0	$-u_1$	u_3	$-e_2$	0	e_3	0
u_3	$2u_3$	$2u_2$	0	$-e_1$	$-e_3$	0	0
u_4	0	0	0	0	0	0	0

3.

$\bar{\mathfrak{g}} = \mathfrak{so}(3,1) \times \mathbb{R}, \;\; \mathfrak{g} = \mathfrak{so}(2,1)$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	$-e_2$	$-e_1$	0
u_2	0	$-u_1$	u_3	e_2	0	$-e_3$	0
u_3	$2u_3$	$2u_2$	0	e_1	e_3	0	0
u_4	0	0	0	0	0	0	0

4.

$\bar{\mathfrak{g}} = (\mathfrak{so}(2,1) \ltimes \mathbb{R}^3) \times \mathbb{R}, \;\; \mathfrak{g} = \mathfrak{so}(2,1)$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	0	0	0
u_2	0	$-u_1$	u_3	0	0	0	0
u_3	$2u_3$	$2u_2$	0	0	0	0	0
u_4	0	0	0	0	0	0	0

3.5^2

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & z & 0 \\ -y & -z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = (\mathfrak{so}(3) \oplus \mathbb{R}\,\mathrm{id}_{\mathbb{R}^3}) \leftthreetimes \mathbb{R}^3,\ \ \mathfrak{g} = \mathfrak{so}(3)$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	$-u_2$	u_1	0	0
e_2	e_3	0	$-e_1$	$-u_3$	0	u_1	0
e_3	$-e_2$	e_1	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	0	0	u_1
u_2	$-u_1$	0	u_3	0	0	0	u_2
u_3	0	$-u_1$	$-u_2$	0	0	0	u_3
u_4	0	0	0	$-u_1$	$-u_2$	$-u_3$	0

2.

$\bar{\mathfrak{g}} = \mathfrak{so}(3,1) \times \mathbb{R},\ \ \mathfrak{g} = \mathfrak{so}(3)$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	$-u_2$	u_1	0	0
e_2	e_3	0	$-e_1$	$-u_3$	0	u_1	0
e_3	$-e_2$	e_1	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	e_1	e_2	0
u_2	$-u_1$	0	u_3	$-e_1$	0	e_3	0
u_3	0	$-u_1$	$-u_2$	$-e_2$	$-e_3$	0	0
u_4	0	0	0	0	0	0	0

3.

$\bar{\mathfrak{g}} = \mathfrak{so}(4) \times \mathbb{R},\ \ \mathfrak{g} = \mathfrak{so}(3)$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	$-u_2$	u_1	0	0
e_2	e_3	0	$-e_1$	$-u_3$	0	u_1	0
e_3	$-e_2$	e_1	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	$-e_1$	$-e_2$	0
u_2	$-u_1$	0	u_3	e_1	0	$-e_3$	0
u_3	0	$-u_1$	$-u_2$	e_2	e_3	0	0
u_4	0	0	0	0	0	0	0

4.

$\bar{\mathfrak{g}} = (\mathfrak{so}(3) \ltimes \mathbb{R}^3) \times \mathbb{R}, \ \ \mathfrak{g} = \mathfrak{so}(3)$

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	$-u_2$	u_1	0	0
e_2	e_3	0	$-e_1$	$-u_3$	0	u_1	0
e_3	$-e_2$	e_1	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	0	0	0
u_2	$-u_1$	0	u_3	0	0	0	0
u_3	0	$-u_1$	$-u_2$	0	0	0	0
u_4	0	0	0	0	0	0	0

4.1^1

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & z & 0 & t \\ 0 & y & -t & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix} \middle| \, x, y, z, t \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \ \ \mathfrak{g} = 4.1^1$

[,]	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	e_3	e_4	u_1	0	$-u_3$	0
e_2	0	0	$-e_3$	e_4	0	u_2	0	$-u_4$
e_3	$-e_3$	e_3	0	0	0	u_1	$-u_4$	0
e_4	$-e_4$	$-e_4$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0	0
u_2	0	$-u_2$	$-u_1$	0	0	0	0	0
u_3	u_3	0	u_4	u_2	0	0	0	0
u_4	0	u_4	0	$-u_1$	0	0	0	0

4.1^2

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & z & 0 & -t \\ 0 & 0 & -z & -y \\ 0 & 0 & -x & 0 \\ 0 & y & t & 0 \end{pmatrix} \middle| \, x, y, z, t \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \ \ \mathfrak{g} = 4.1^2$

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	e_3	e_4	u_1	0	$-u_3$	0
e_2	0	0	$-e_4$	e_3	0	u_4	0	$-u_2$
e_3	$-e_3$	e_4	0	0	0	u_1	$-u_2$	0
e_4	$-e_4$	$-e_3$	0	0	0	0	u_4	$-u_1$
u_1	$-u_1$	0	0	0	0	0	0	0
u_2	0	$-u_4$	$-u_1$	0	0	0	0	0
u_3	u_3	0	u_2	$-u_4$	0	0	0	0
u_4	0	u_2	0	u_1	0	0	0	0

4.2[1]

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 & 0 \\ z & t & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & -t \end{pmatrix} \right| x, y, z, t \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{sl}(3,\mathbb{R}),\ \ \mathfrak{g} = \mathfrak{gl}(2,\mathbb{R})$

[,]	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	e_1+3e_2	$2e_3$
u_2	u_2	$-u_2$	$-u_1$	0	0	0	$2e_4$	$-e_1+3e_2$
u_3	u_3	u_3	u_4	0	$-e_1-3e_2$	$-2e_4$	0	0
u_4	$-u_4$	u_4	0	u_3	$-2e_3$	e_1-3e_2	0	0

2.

$\bar{\mathfrak{g}} = \mathfrak{gl}(2,\mathbb{R}) \ltimes (V \oplus V^*)$, where $V = \mathbb{R}^2$ is natural $\mathfrak{gl}(2,\mathbb{R})$-module,
$\mathfrak{g} = \mathfrak{gl}(2,\mathbb{R})$

[,]	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	0	0
u_2	u_2	$-u_2$	$-u_1$	0	0	0	0	0
u_3	u_3	u_3	u_4	0	0	0	0	0
u_4	$-u_4$	u_4	0	u_3	0	0	0	0

4.2^2

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & y & -x & -z \\ -y & 0 & -z & -t \\ x & z & 0 & y \\ z & t & -y & 0 \end{pmatrix} \middle|\ x,y,z,t \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{su}(3),\ \ \mathfrak{g} = \mathfrak{u}(2)$

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_4$	$-2e_3$	u_3	$-u_4$	$-u_1$	u_2
e_2	0	0	0	0	u_3	u_4	$-u_1$	$-u_2$
e_3	$-2e_4$	0	0	$2e_1$	$-u_2$	u_1	$-u_4$	u_3
e_4	$2e_3$	0	$-2e_1$	0	u_4	u_3	$-u_2$	$-u_1$
u_1	$-u_3$	$-u_3$	u_2	$-u_4$	0	$-e_3$	e_1+3e_2	e_4
u_2	u_4	$-u_4$	$-u_1$	$-u_3$	e_3	0	e_4	$-e_1+3e_2$
u_3	u_1	u_1	u_4	u_2	$-e_1-3e_2$	$-e_4$	0	$-e_3$
u_4	$-u_2$	u_2	$-u_3$	u_1	$-e_4$	e_1-3e_2	e_3	0

2.

$\bar{\mathfrak{g}} = \mathfrak{su}(2,1), \ \ \mathfrak{g} = \mathfrak{u}(2)$

[,]	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_4$	$-2e_3$	u_3	$-u_4$	$-u_1$	u_2
e_2	0	0	0	0	u_3	u_4	$-u_1$	$-u_2$
e_3	$-2e_4$	0	0	$2e_1$	$-u_2$	u_1	$-u_4$	u_3
e_4	$2e_3$	0	$-2e_1$	0	u_4	u_3	$-u_2$	$-u_1$
u_1	$-u_3$	$-u_3$	u_2	$-u_4$	0	e_3	$-e_1 - 3e_2$	$-e_4$
u_2	u_4	$-u_4$	$-u_1$	$-u_3$	$-e_3$	0	$-e_4$	$e_1 - 3e_2$
u_3	u_1	u_1	u_4	u_2	$e_1 + 3e_2$	e_4	0	e_3
u_4	$-u_2$	u_2	$-u_3$	u_1	e_4	$3e_2 - e_1$	$-e_3$	0

3.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \ \ \mathfrak{g} = 4.2^2 \cong \mathfrak{u}(2)$, $\mathfrak{g}$ acts irreducibly on $\mathbb{R}^4$

[,]	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_4$	$-2e_3$	u_3	$-u_4$	$-u_1$	u_2
e_2	0	0	0	0	u_3	u_4	$-u_1$	$-u_2$
e_3	$-2e_4$	0	0	$2e_1$	$-u_2$	u_1	$-u_4$	u_3
e_4	$2e_3$	0	$-2e_1$	0	u_4	u_3	$-u_2$	$-u_1$
u_1	$-u_3$	$-u_3$	u_2	$-u_4$	0	0	0	0
u_2	u_4	$-u_4$	$-u_1$	$-u_3$	0	0	0	0
u_3	u_1	u_1	u_4	u_2	0	0	0	0
u_4	$-u_2$	u_2	$-u_3$	u_1	0	0	0	0

4.2^3

$$\mathfrak{g} = \left\{ \left(\begin{array}{cccc} x & y & 0 & t \\ z & -x & -t & 0 \\ 0 & t & -x & -z \\ -t & 0 & -y & x \end{array} \right) \middle| \ x, y, z, t \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{su}(2,1), \ \ \mathfrak{g} = \mathfrak{gl}(2,\mathbb{R})$

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$-u_4$	u_3	$-u_2$	u_1
e_3	$-2e_3$	0	0	e_1	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	u_4	0	$-u_2$	0	$3e_2$	e_1	$2e_3$
u_2	u_2	$-u_3$	$-u_1$	0	$-3e_2$	0	$2e_4$	$-e_1$
u_3	u_3	u_2	u_4	0	$-e_1$	$-2e_4$	0	$3e_2$
u_4	$-u_4$	$-u_1$	0	u_3	$-2e_3$	e_1	$-3e_2$	0

2.

$$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \ \ \mathfrak{g} = 4.2^3 \cong \mathfrak{gl}(2,\mathbb{R}), \ \mathfrak{g} \text{ acts irreducibly on } \mathbb{R}^4$$

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$-u_4$	u_3	$-u_2$	u_1
e_3	$-2e_3$	0	0	e_1	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	u_4	0	$-u_2$	0	0	0	0
u_2	u_2	$-u_3$	$-u_1$	0	0	0	0	0
u_3	u_3	u_2	u_4	0	0	0	0	0
u_4	$-u_4$	$-u_1$	0	u_3	0	0	0	0

4.3[1]

$$\mathfrak{g} = \left\{ \left(\begin{array}{cccc} x & y & 0 & t \\ z & -x & -t & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{array} \right) \middle| \; x, y, z, t \in \mathbb{R} \right\}$$

1.

$$\bar{\mathfrak{g}} = \left\{ \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ 0 & z & -x & 0 & 0 \\ 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & z & -x \end{array} \right) \middle| \; x, y, z \in \mathbb{R} \right\} \ltimes \tilde{\mathfrak{n}}_5,$$

$$\mathfrak{g} = \left\{ \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ 0 & z & -x & 0 & 0 \\ 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & z & -x \end{array} \right) \middle| \; x, y, z \in \mathbb{R} \right\} \times \mathbb{R}h$$

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	0	u_2	0	0	$-u_3$
e_4	0	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_4	0	u_2	0	0	0	e_4
u_4	$-u_4$	0	u_3	$-u_1$	0	0	$-e_4$	0

2.

$$\bar{\mathfrak{g}} = \mathfrak{g} \leftthreetimes \mathbb{R}^4, \;\; \mathfrak{g} = 4.3^1 \cong \mathfrak{gl}(2, \mathbb{R})$$

[,]	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	0	u_2	0	0	$-u_3$
e_4	0	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_4	0	u_2	0	0	0	0
u_4	$-u_4$	0	u_3	$-u_1$	0	0	0	0

5.1[1]

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 & u \\ z & t & -u & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & -t \end{pmatrix} \right| x,y,z,t,u \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^4, \;\; \mathfrak{g} = 5.1^1$

[,]	e_1	e_2	e_3	e_4	e_5	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_2$	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_3	u_4	0	u_2	0	0	0	0
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	0	0	0

6.1[1]

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 & u \\ z & t & -u & 0 \\ 0 & v & -x & -z \\ -v & 0 & -y & -t \end{pmatrix} \right| x,y,z,t,u,v \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{so}(3,2), \;\; \mathfrak{g} = \mathfrak{so}(2,2)$

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	$-2e_6$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	$-e_2$	0	0	$-u_2$	u_1
e_6	0	$2e_6$	0	0	e_2	0	$-u_4$	u_3	0	0
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	u_4	0	$2e_5$	e_1+e_2	$2e_3$
u_2	u_2	$-u_2$	$-u_1$	0	0	$-u_3$	$-2e_5$	0	$2e_4$	$-e_1+e_2$
u_3	u_3	u_3	u_4	0	u_2	0	$-e_1-e_2$	$-2e_4$	0	$2e_6$
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	$-2e_3$	e_1-e_2	$-2e_6$	0

2.

$\bar{\mathfrak{g}} = \mathfrak{so}(2,2) \rightthreetimes \mathbb{R}^4, \;\; \mathfrak{g} = \mathfrak{so}(2,2)$

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	$-2e_6$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	$-e_2$	0	0	$-u_2$	u_1
e_6	0	$2e_6$	0	0	e_2	0	$-u_4$	u_3	0	0
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	u_4	0	0	0	0
u_2	u_2	$-u_2$	$-u_1$	0	0	$-u_3$	0	0	0	0
u_3	u_3	u_3	u_4	0	u_2	0	0	0	0	0
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	0	0	0	0

6.1^2

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & x & y & z \\ -x & 0 & t & u \\ -y & -t & 0 & v \\ -z & -u & -v & 0 \end{pmatrix} \middle| \; x, y, z, t, u, v \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{so}(4,1), \;\; \mathfrak{g} = \mathfrak{so}(4)$

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$-e_4$	$-e_5$	e_2	e_3	0	$-u_2$	u_1	0	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3	$-u_3$	0	u_1	0
e_3	e_5	e_6	0	0	$-e_1$	$-e_2$	$-u_4$	0	0	u_1
e_4	$-e_2$	e_1	0	0	$-e_6$	e_5	0	$-u_3$	u_2	0
e_5	$-e_3$	0	e_1	e_6	0	$-e_4$	0	$-u_4$	0	u_2
e_6	0	$-e_3$	e_2	$-e_5$	e_4	0	0	0	$-u_4$	u_3
u_1	u_2	u_3	u_4	0	0	0	0	e_1	e_2	e_3
u_2	$-u_1$	0	0	u_3	u_4	0	$-e_1$	0	e_4	e_5
u_3	0	$-u_1$	0	$-u_2$	0	u_4	$-e_2$	$-e_4$	0	e_6
u_4	0	0	$-u_1$	0	$-u_2$	$-u_3$	$-e_3$	$-e_5$	$-e_6$	0

2.

$\bar{\mathfrak{g}} = \mathfrak{so}(5), \;\; \mathfrak{g} = \mathfrak{so}(4)$

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$-e_4$	$-e_5$	e_2	e_3	0	$-u_2$	u_1	0	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3	$-u_3$	0	u_1	0
e_3	e_5	e_6	0	0	$-e_1$	$-e_2$	$-u_4$	0	0	u_1
e_4	$-e_2$	e_1	0	0	$-e_6$	e_5	0	$-u_3$	u_2	0
e_5	$-e_3$	0	e_1	e_6	0	$-e_4$	0	$-u_4$	0	u_2
e_6	0	$-e_3$	e_2	$-e_5$	e_4	0	0	0	$-u_4$	u_3
u_1	u_2	u_3	u_4	0	0	0	0	$-e_1$	$-e_2$	$-e_3$
u_2	$-u_1$	0	0	u_3	u_4	0	e_1	0	$-e_4$	$-e_5$
u_3	0	$-u_1$	0	$-u_2$	0	u_4	e_2	e_4	0	$-e_6$
u_4	0	0	$-u_1$	0	$-u_2$	$-u_3$	e_3	e_5	e_6	0

3.

$\bar{\mathfrak{g}} = \mathfrak{so}(4) \ltimes \mathbb{R}^4, \;\; \mathfrak{g} = \mathfrak{so}(4)$

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$-e_4$	$-e_5$	e_2	e_3	0	$-u_2$	u_1	0	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3	$-u_3$	0	u_1	0
e_3	e_5	e_6	0	0	$-e_1$	$-e_2$	$-u_4$	0	0	u_1
e_4	$-e_2$	e_1	0	0	$-e_6$	e_5	0	$-u_3$	u_2	0
e_5	$-e_3$	0	e_1	e_6	0	$-e_4$	0	$-u_4$	0	u_2
e_6	0	$-e_3$	e_2	$-e_5$	e_4	0	0	0	$-u_4$	u_3
u_1	u_2	u_3	u_4	0	0	0	0	0	0	0
u_2	$-u_1$	0	0	u_3	u_4	0	0	0	0	0
u_3	0	$-u_1$	0	$-u_2$	0	u_4	0	0	0	0
u_4	0	0	$-u_1$	0	$-u_2$	$-u_3$	0	0	0	0

6.1^3

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & x & y & z \\ -x & 0 & t & u \\ -y & -t & 0 & v \\ z & u & v & 0 \end{pmatrix} \middle| \; x, y, z, t, u, v \in \mathbb{R} \right\}$$

1.

$\bar{\mathfrak{g}} = \mathfrak{so}(3,2)$, $\mathfrak{g} = \mathfrak{so}(3,1)$

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$-e_4$	$-e_5$	e_2	e_3	0	$-u_2$	u_1	0	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3	$-u_3$	0	u_1	0
e_3	e_5	e_6	0	0	e_1	e_2	u_4	0	0	u_1
e_4	$-e_2$	e_1	0	0	$-e_6$	e_5	0	$-u_3$	u_2	0
e_5	$-e_3$	0	$-e_1$	e_6	0	e_4	0	u_4	0	u_2
e_6	0	$-e_3$	$-e_2$	$-e_5$	$-e_4$	0	0	0	u_4	u_3
u_1	u_2	u_3	$-u_4$	0	0	0	0	e_1	e_2	$-e_3$
u_2	$-u_1$	0	0	u_3	$-u_4$	0	$-e_1$	0	e_4	$-e_5$
u_3	0	$-u_1$	0	$-u_2$	0	$-u_4$	$-e_2$	e_4	0	$-e_6$
u_4	0	0	$-u_1$	0	$-u_2$	$-u_3$	e_3	e_5	e_6	0

2.

$\bar{\mathfrak{g}} = \mathfrak{so}(4,1)$, $\mathfrak{g} = \mathfrak{so}(3,1)$

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$-e_4$	$-e_5$	e_2	e_3	0	$-u_2$	u_1	0	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3	$-u_3$	0	u_1	0
e_3	e_5	e_6	0	0	e_1	e_2	u_4	0	0	u_1
e_4	$-e_2$	e_1	0	0	$-e_6$	e_5	0	$-u_3$	u_2	0
e_5	$-e_3$	0	$-e_1$	e_6	0	e_4	0	u_4	0	u_2
e_6	0	$-e_3$	$-e_2$	$-e_5$	$-e_4$	0	0	0	u_4	u_3
u_1	u_2	u_3	$-u_4$	0	0	0	0	$-e_1$	$-e_2$	e_3
u_2	$-u_1$	0	0	u_3	$-u_4$	0	e_1	0	$-e_4$	e_5
u_3	0	$-u_1$	0	$-u_2$	0	$-u_4$	e_2	$-e_4$	0	e_6
u_4	0	0	$-u_1$	0	$-u_2$	$-u_3$	$-e_3$	$-e_5$	$-e_6$	0

3.

$$\bar{\mathfrak{g}} = \mathfrak{so}(3,1) \ltimes \mathbb{R}^4, \quad \mathfrak{g} = \mathfrak{so}(3,1)$$

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$-e_4$	$-e_5$	e_2	e_3	0	$-u_2$	u_1	0	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3	$-u_3$	0	u_1	0
e_3	e_5	e_6	0	0	e_1	e_2	u_4	0	0	u_1
e_4	$-e_2$	e_1	0	0	$-e_6$	e_5	0	$-u_3$	u_2	0
e_5	$-e_3$	0	$-e_1$	e_6	0	e_4	0	u_4	0	u_2
e_6	0	$-e_3$	$-e_2$	$-e_5$	$-e_4$	0	0	0	u_4	u_3
u_1	u_2	u_3	$-u_4$	0	0	0	0	0	0	0
u_2	$-u_1$	0	0	u_3	$-u_4$	0	0	0	0	0
u_3	0	$-u_1$	0	$-u_2$	0	$-u_4$	0	0	0	0
u_4	0	0	$-u_1$	0	$-u_2$	$-u_3$	0	0	0	0

CHAPTER II

EINSTEIN–MAXWELL EQUATION ON FOUR-DIMENSIONAL HOMOGENEOUS SPACES

Preliminaries:

Let $(\overline{G}, M)$ be a homogeneous space with an invariant pseudo-Riemannian metric g. Let ω be an invariant differential 2-form on M which is closed $(d\omega = 0)$ and coclosed $(d(*\omega) = 0)$. The Einstein–Maxwell equation on the homogeneous space $(\overline{G}, M)$ has the form:

$$r - \lambda g = m_\omega, \tag{1}$$

where λ is an arbitrary real number, r is the Ricci tensor of g, m_ω is an invariant smooth tensor field of type (0,2) on M defined by $m_\omega = \omega \circ g^{-1} \circ \omega$, where g, ω, m_ω are identified with the corresponding mappings $TM \to T^*M$.

Equation (1) is called the Einstein equation if $m_\omega \equiv 0$.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of Lie algebras corresponding to the homogeneous space $(\overline{G}, M)$. By $\mathfrak{m}$ denote the factor space $\bar{\mathfrak{g}}/\mathfrak{g}$ endowed with the natural structure of $\mathfrak{g}$-module, and let $x_\mathfrak{m} \in \mathfrak{m}$ denote the image of the element $x \in \mathfrak{g}$ under the natural projection $\pi \colon \bar{\mathfrak{g}} \to \mathfrak{m}$. Then, in terms of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$, the Einstein–Maxwell equation on the homogeneous space $(\overline{G}, M)$ will have the form:

$$R - \lambda B = M_\Omega,$$

where B is an invariant nondegenerate symmetric bilinear form on $\mathfrak{m}$, Ω is an invariant skew-symmetric bilinear form on $\mathfrak{m}$ which is closed $(d\Omega = 0)$ and coclosed $(d(*\Omega) = 0)$, R is the invariant symmetric bilinear form on $\mathfrak{m}$ corresponding to the Ricci tensor, λ is an arbitrary real number, and M_Ω is the invariant symmetric bilinear form on $\mathfrak{m}$ corresponding to m_ω.

Now we identify $\mathfrak{m}$ with some (not necessarily $\mathfrak{g}$-invariant) subspace of $\bar{\mathfrak{g}}$ complementary to $\mathfrak{g}$. Let $\mathcal{E} = \{e_1, ..., e_n, u_1, u_2, u_3, u_4\}$ be a basis of $\bar{\mathfrak{g}}$, where $\{e_1, ..., e_n\}$ is a basis of $\mathfrak{g}$ and $U = \{u_1, u_2, u_3, u_4\}$ is a basis of $\mathfrak{m}$. Then we can identify any bilinear form $\widetilde{B}$ on $\mathfrak{m}$ with the matrix of $\widetilde{B}$ with respect to the basis U. We divide the solution of the Einstein–Maxwell equation on the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ into the following parts:

1. We find B and Ω. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ uniquely defines the isotropic representation

$$\rho \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m}), \qquad \rho(x)(m) = [x, m]_\mathfrak{m} \quad \text{for all } x \in \mathfrak{g},\ m \in \mathfrak{m}.$$

Let $\widetilde{B} \in \mathrm{Bil}(\mathfrak{m})$ be an arbitrary bilinear form on $\mathfrak{m}$. Then the invariance condition for $\widetilde{B}$ has the form: ${}^tX\widetilde{B} + \widetilde{B}X = 0$ for all $X \in \rho(\mathfrak{g})$. Verifying symmetry (skew-symmetry) condition, we obtain $B(\Omega)$.

2. Ω is closed, i.e., $d\Omega = 0$. It means that ([F])

$$d\Omega(x,y,z) = -\Omega([x,y]_{\mathfrak{m}}, z) + \Omega([x,z]_{\mathfrak{m}}, y) - \Omega([y,z]_{\mathfrak{m}}, x) = 0 \quad \text{for all } x,y,z \in \mathfrak{m}.$$

3. Ω is coclosed, i.e., $(\dot{*}\,\Omega) = 0$, where $*$ is the Hodge operator corresponding to B.

4. We compute $M_\Omega = \Omega B^{-1} \Omega$.

5. There is a one-to-one correspondence between the invariant linear connections on $(\overline{G}, M)$ and the homomorphisms of $\mathfrak{g}$-modules $\Lambda\colon \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda(x)(y_{\mathfrak{m}}) = [x,y]_{\mathfrak{m}}$ for all $x \in \mathfrak{g}$, $y \in \bar{\mathfrak{g}}$. It follows that $\Lambda(x) = \rho(x)$ for all $x \in \mathfrak{g}$. The invariant nondegenerate symmetric bilinear form B uniquely defines the invariant linear Levi–Civita connection $\Lambda\colon \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ in the following way. Let $v\colon \bar{\mathfrak{g}} \times \bar{\mathfrak{g}} \to \mathfrak{m}$ be the $\mathfrak{g}$-invariant symmetric mapping uniquely determined by the equality

$$2B(v(x,y), z_{\mathfrak{m}}) = B(x_{\mathfrak{m}}, [z,y]_{\mathfrak{m}}) + B([z,x]_{\mathfrak{m}}, y_{\mathfrak{m}}) \quad \text{for all } x,y,z \in \bar{\mathfrak{g}}.$$

Then $\Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x,y]_{\mathfrak{m}} + v(x,y) \quad$ for all $x,y \in \bar{\mathfrak{g}}$.

6. To the curvature tensor of the connection Λ we assign the mapping $K\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ such that $K(x,y) = [\Lambda(x), \Lambda(y)] - \Lambda([x,y]) \quad$ for all $x,y \in \mathfrak{m}$.

7. To the Ricci tensor of the connection Λ we assign the symmetric bilinear form $R\colon \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ such that

$$R(u_i, u_j) = \sum_{l=1}^{4} K_{li}(u_l, u_j), \quad u_i, u_j \in U,\ i,j = \overline{1,4}.$$

8. We find the solutions. The solution of the Einstein–Maxwell equation is any triple (B, Ω, λ) satisfying the equality $R - \lambda B = M_\Omega$.

Remarks.

1. Note that all expressions are linear in all arguments. Therefore, in order to ensure that these conditions are satisfied for all arguments, we must only check that they hold for basis vectors.

2. By a, b, c, etc. we denote the entries of B, and by α, β, γ, etc. the entries of Ω. Unless otherwise stated, we assume that the entries of these matrices run through $\mathbb{R}$. Any additional conditions appear just after the matrices.

3. The matrix B must satisfy the condition $\det B \neq 0$.

4. For more details about parts 5–7, see [K-N].

Example. Consider the pair

1.1^1.2.

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y-x & 0 \\ 0 & 0 & py \end{pmatrix} \middle|\, x,y \in \mathbb{R},\ p \in \mathbb{R} \right\} \ltimes \mathbb{R}^3, \ \mathfrak{g} = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle|\, x \in \mathbb{R} \right\}$$

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	pu_2
u_3	u_3	0	0	0	u_3
u_4	0	0	$-pu_2$	$-u_3$	0

1. Checking invariance condition for arbitrary bilinear form $\widetilde{B}$ on $\mathfrak{m}$ we obtain that

$$\widetilde{B} = \begin{pmatrix} 0 & 0 & b_1 & 0 \\ 0 & b_2 & 0 & b_3 \\ b_4 & 0 & 0 & 0 \\ 0 & b_5 & 0 & b_6 \end{pmatrix}. \quad \text{Therefore, } B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \ \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}.$$

2. $d\Omega(u_3, u_4, u_1) = -\Omega([u_3, u_4], u_1) + \Omega([u_3, u_1], u_4) - \Omega([u_4, u_1], u_3) = \alpha = 0 \Rightarrow$

$\alpha = 0$. $d\Omega(u_i, u_j, u_k) = 0$, $i, j, k = \overline{1,4}$. It follows that Ω is closed $\iff \alpha = 0$.

3. $$*\Omega = \begin{pmatrix} 0 & 0 & -a^2\beta & 0 \\ 0 & 0 & 0 & (bd-c^2)\alpha \\ a^2\beta & 0 & 0 & 0 \\ 0 & -(bd-c^2)\alpha & 0 & 0 \end{pmatrix}.$$

By conditions $d(*\Omega) = 0$ and $\alpha = 0$ we obtain that Ω is closed and coclosed $\iff \Omega = 0$.

4. $M_\Omega = \Omega B^{-1} \Omega = 0$.

5. The mapping $v|_{\mathfrak{m}} \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ has the form:

$v(u_i, u_j)$	u_1	u_2	u_3	u_4
u_1	0	0	$\frac{ac}{2(bd-c^2)}u_2 - \frac{ab}{2(bd-c^2)}u_4$	$\frac{1}{2}u_1$
u_2	0	$\frac{pbc}{bd-c^2}u_2 - \frac{pb^2}{bd-c^2}u_4$	0	$\frac{p(bd+c^2)}{2(bd-c^2)}u_2 - \frac{pbc}{bd-c^2}u_4$
u_3	$\frac{ac}{2(bd-c^2)}u_2 - \frac{ab}{2(bd-c^2)}u_4$	0	0	0
u_4	$\frac{1}{2}u_1$	$\frac{p(bd+c^2)}{2(bd-c^2)}u_2 - \frac{pbc}{bd-c^2}u_4$	0	$\frac{pdc}{bd-c^2}u_2 - \frac{pc^2}{bd-c^2}u_4$

It follows that the invariant linear Levi–Civita connection Λ has the form:

$$\Lambda(e_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda(u_1)=\begin{pmatrix}0&0&0&\frac{1}{2}\\0&0&\frac{ac}{2(bd-c^2)}&0\\0&0&0&0\\0&0&-\frac{ab}{2(bd-c^2)}&0\end{pmatrix},\quad \Lambda(u_2)=\begin{pmatrix}0&0&0&0\\0&\frac{pbc}{bd-c^2}&0&\frac{pbd}{bd-c^2}\\0&0&0&0\\0&-\frac{pb^2}{bd-c^2}&0&-\frac{pbc}{bd-c^2}\end{pmatrix},$$

$$\Lambda(u_3)=\begin{pmatrix}0&0&0&0\\\frac{ac}{2(bd-c^2)}&0&0&0\\0&0&0&\frac{1}{2}\\-\frac{ab}{2(bd-c^2)}&0&0&0\end{pmatrix},\quad \Lambda(u_4)=\begin{pmatrix}\frac{1}{2}&0&0&0\\0&\frac{pc^2}{bd-c^2}&0&\frac{pdc}{bd-c^2}\\0&0&-\frac{1}{2}&0\\0&-\frac{pbc}{bd-c^2}&0&-\frac{pc^2}{bd-c^2}\end{pmatrix}.$$

6. The curvature tensor of the invariant linear Levi–Civita connection Λ has the form:

$$K(u_1,u_2)=\begin{pmatrix} 0 & -\frac{pb^2}{2(bd-c^2)} & 0 & -\frac{pbc}{2(bd-c^2)} \\ 0 & 0 & \frac{pab}{2(bd-c^2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K(u_1,u_3)=\begin{pmatrix} -\frac{ab}{4(bd-c^2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{ab}{4(bd-c^2)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K(u_1,u_4)=\begin{pmatrix} 0 & -\frac{pbc}{2(bd-c^2)} & 0 & -\frac{pc^2}{2(bd-c^2)}-\frac{1}{4} \\ 0 & 0 & \frac{(2p-1)ac}{4(bd-c^2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{ab}{4(bd-c^2)} & 0 \end{pmatrix},$$

$$K(u_2,u_3)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{pab}{2(bd-c^2)} & 0 & 0 & 0 \\ 0 & \frac{pb^2}{2(bd-c^2)} & 0 & \frac{pbc}{2(bd-c^2)} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K(u_2,u_4)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{p^2bc}{bd-c^2} & 0 & -\frac{p^2bd}{bd-c^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{p^2b^2}{bd-c^2} & 0 & \frac{p^2bc}{bd-c^2} \end{pmatrix},$$

$$K(u_3,u_4)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{(2p-1)ac}{4(bd-c^2)} & 0 & 0 & 0 \\ 0 & -\frac{pbc}{2(bd-c^2)} & 0 & -\frac{pc^2}{2(bd-c^2)}-\frac{1}{4} \\ \frac{ab}{4(bd-c^2)} & 0 & 0 & 0 \end{pmatrix}.$$

7. The Ricci tensor of the invariant linear Levi–Civita connection Λ has the form:

$$R=\begin{pmatrix} 0 & 0 & -\frac{ab(p+1)}{2(bd-c^2)} & 0 \\ 0 & -\frac{p(p+1)b^2}{bd-c^2} & 0 & -\frac{p(p+1)bc}{bd-c^2} \\ -\frac{ab(p+1)}{2(bd-c^2)} & 0 & 0 & 0 \\ 0 & -\frac{p(p+1)bc}{bd-c^2} & 0 & -\frac{(p^2bd+pc^2)}{bd-c^2}-\frac{1}{2} \end{pmatrix}.$$

8. The Einstein–Maxwell equation has the form: $R-\lambda B=M_\Omega$.

$$\begin{pmatrix} 0 & 0 & -\frac{ab(p+1)}{2(bd-c^2)} & 0 \\ 0 & -\frac{p(p+1)b^2}{bd-c^2} & 0 & -\frac{p(p+1)bc}{bd-c^2} \\ -\frac{ab(p+1)}{2(bd-c^2)} & 0 & 0 & 0 \\ 0 & -\frac{p(p+1)bc}{bd-c^2} & 0 & -\frac{(p^2bd+pc^2)}{bd-c^2} - \frac{1}{2} \end{pmatrix} - \lambda \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix} = 0.$$

It follows that any solution of the Einstein–Maxwell equation on the pair $1.1^1.2$ has the form:

$p = \frac{1}{2}$

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = 0, \quad \lambda = -\frac{3b}{4(bd-c^2)}.$$

For other values of p the Einstein–Maxwell equation has no solutions.

1. Solutions of Einstein–Maxwell equation on pseudo-Riemannian pairs of codimension 4

Proposition $1.1^1.1$. *Any solution of the Einstein–Maxwell equation on the pair* $1.1^1.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix},$$

where $a^2 = c^2 - bd$,

$$\Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{3b - 2\alpha^2}{2a^2}.$$

Proposition $1.1^1.2$. *Any solution of the Einstein–Maxwell equation on the pair* $1.1^1.2$ *has the form:*

$p = \frac{1}{2}$

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{3b}{4(bd-c^2)}.$$

For other values of p *the Einstein–Maxwell equation has no solutions.*

Proposition $1.1^1.3$. *The Einstein–Maxwell equation has no solutions on the pair* $1.1^1.3$.

Proposition 1.1^1.4. *The Einstein–Maxwell equation has no solutions on the pair* 1.1^1.4.

Proposition 1.1^1.5. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^1.5 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2 - b}{bd - c^2},$$

where $\alpha^2 = \frac{b-\beta^2}{bd-c^2} a^2 - a$.

Proposition 1.1^1.6. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^1.6 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2 - b}{bd - c^2},$$

where $\alpha^2 = \frac{b-\beta^2}{bd-c^2} a^2$.

Proposition 1.1^1.7. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^1.7 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2}{bd - c^2},$$

where $\alpha^2 = -\frac{\beta^2 a^2}{bd-c^2} - a$.

Proposition 1.1^1.8. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^1.8 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\beta^2}{b^2},$$

where $\alpha^2 = \frac{\beta^2 a^2}{b^2} + 3a$.

Proposition 1.1^1.9. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^1.9 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\beta^2}{b^2},$$

where $\alpha^2 = \frac{\beta^2 a^2}{b^2}$.

Proposition 1.1[1].10. *Any solution of the Einstein–Maxwell equation on the pair* 1.1[1].10 *has the form:*

$\lambda = 0$

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2}{bd - c^2},$$

where $\alpha^2 = -\frac{\beta^2 a^2}{bd-c^2}$.

$\lambda = 1$

$$B = \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & C \\ -C^t & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} \in \mathbb{R},$$

where $A \in GL(2,\mathbb{R})$, $C \in Mat(2,\mathbb{R})$, *such that* $-\boldsymbol{\lambda} A = CA^{-1}C$.

$\lambda \in]0,1[$

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\beta^2}{b^2},$$

where $\alpha^2 = \frac{\beta^2 a^2}{b^2}$.

Proposition 1.1[2].1. *Any solution of the Einstein–Maxwell equation on the pair* 1.1[2].1 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix},$$

where $4a^2 = bd - c^2$,

$$\Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -2\alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & 2\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{2\alpha^2 - 3b}{2a^2}.$$

Proposition 1.1[2].2. *Any solution of the Einstein–Maxwell equation on the pair* 1.1[2].2 *has the form:*

$p = 1$

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{3b}{bd - c^2}.$$

For other values of p *the Einstein–Maxwell equation has no solutions.*

Proposition 1.1[2].3. *The Einstein–Maxwell equation has no solutions on the pair* 1.1[2].3.

Proposition 1.1^2.4. *The Einstein–Maxwell equation has no solutions on the pair* 1.1^2.4.

Proposition 1.1^2.5. *The Einstein–Maxwell equation has no solutions on the pair* 1.1^2.5.

Proposition 1.1^2.6. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^2.6 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2 - b}{bd - c^2},$$

where $\alpha^2 = \frac{\beta^2 - b}{bd-c^2} a^2 - a$.

Proposition 1.1^2.7. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^2.7 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2 - b}{bd - c^2},$$

where $\alpha^2 = \frac{\beta^2 - b}{bd-c^2} a^2 + a$.

Proposition 1.1^2.8. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^2.8 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2 - b}{bd - c^2},$$

where $\alpha^2 = \frac{\beta^2 - b}{bd-c^2} a^2$.

Proposition 1.1^2.9. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^2.9 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2}{bd - c^2},$$

where $\alpha^2 = \frac{\beta^2 a^2}{bd-c^2} - a$.

Proposition 1.1^2.10. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^2.10 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2}{bd - c^2},$$

where $\alpha^2 = \frac{\beta^2 a^2}{bd-c^2} + a$.

Proposition 1.1^2.11. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^2.11 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2}{b^2},$$

where $\alpha^2 = \frac{\beta^2 a^2}{b^2} + 6a$.

Proposition 1.1^2.12. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^2.12 *has the form:*

$\lambda = 0$

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2}{bd - c^2},$$

where $\alpha^2 = \frac{\beta^2 a^2}{bd - c^2}$.

$\lambda = 1$

$$B = \begin{pmatrix} a & b & 0 & d \\ b & c & -d & 0 \\ 0 & -d & a & b \\ d & 0 & b & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & \gamma & \alpha & \delta \\ -\gamma & 0 & \delta & \beta \\ -\alpha & -\delta & 0 & \gamma \\ -\delta & -\beta & -\gamma & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} \in \mathbb{R},$$

such that $-\boldsymbol{\lambda} B = M_\Omega$, *where* $M_\Omega = \Omega B^{-1} \Omega$.

$\lambda \in]0,1[$

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2}{b^2},$$

where $\alpha^2 = \frac{\beta^2 a^2}{b^2}$.

Proposition 1.1^3.1. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^3.1 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition 1.1^4.1. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^4.1 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & a & 0 \\ 0 & b & 0 & 0 \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition 1.1^5.1. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^5.1 *has the form:*

$$B=\begin{pmatrix}0&0&a&b\\0&0&b&-a\\a&b&0&0\\b&-a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\alpha&\beta\\0&0&\beta&-\alpha\\-\alpha&-\beta&0&0\\-\beta&\alpha&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{\alpha^2-\beta^2}{b^2-a^2},$$

where $a^2\neq b^2$, $\alpha\beta=\frac{\beta^2-\alpha^2}{b^2-a^2}ab$, *or*

$$B=\begin{pmatrix}0&0&a&\varepsilon a\\0&0&\varepsilon a&-a\\a&\varepsilon a&0&0\\\varepsilon a&-a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\alpha&\delta\alpha\\0&0&\delta\alpha&-\alpha\\-\alpha&-\delta\alpha&0&0\\-\delta\alpha&\alpha&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\delta\alpha^2}{\varepsilon a^2},$$

where $\varepsilon\in\{-1,1\}$, $\delta\in\{-1,1\}$.

Proposition 1.1^6.1. *Any solution of the Einstein–Maxwell equation on the pair* 1.1^6.1 *has the form:*

$$B=\begin{pmatrix}0&0&a&b\\0&0&-b&a\\a&-b&0&0\\b&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\alpha&\beta\\0&0&-\beta&\alpha\\-\alpha&\beta&0&0\\-\beta&-\alpha&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{\alpha^2-\beta^2}{b^2-a^2},$$

where $a^2\neq b^2$, $\alpha\beta=\frac{\beta^2-\alpha^2}{b^2-a^2}ab$, *or*

$$B=\begin{pmatrix}0&0&a&\varepsilon a\\0&0&-\varepsilon a&a\\a&-\varepsilon a&0&0\\\varepsilon a&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\alpha&\delta\alpha\\0&0&-\delta\alpha&\alpha\\-\alpha&\delta\alpha&0&0\\-\delta\alpha&-\alpha&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\delta\alpha^2}{\varepsilon a^2},$$

where $\varepsilon\in\{-1,1\}$, $\delta\in\{-1,1\}$.

Proposition 1.2^1.1. *Any solution of the Einstein–Maxwell equation on the pair* 1.2^1.1 *has the form:*

$$B=\begin{pmatrix}0&0&a&0\\0&0&b&a\\a&b&0&0\\0&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\alpha&0\\0&0&\frac{\alpha b}{a}&\alpha\\-\alpha&-\frac{\alpha b}{a}&0&0\\0&-\alpha&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\alpha^2}{a^2},$$

where $\alpha\neq 0$, *or*

$$B=\begin{pmatrix}0&0&a&0\\0&0&b&a\\a&b&0&0\\0&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\beta&0\\0&-\beta&0&0\\0&0&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

Proposition 1.2².1. *Any solution of the Einstein–Maxwell equation on the pair* 1.2².1 *has the form:*

$$B=\begin{pmatrix}0&0&a&0\\0&0&0&a\\a&0&b&0\\0&a&0&b\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&\alpha\\0&0&-\alpha&0\\0&\alpha&0&\frac{\alpha b}{a}\\-\alpha&0&-\frac{\alpha b}{a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{\alpha^2}{a^2},$$

where $\alpha\neq 0$, *or*

$$B=\begin{pmatrix}0&0&a&0\\0&0&0&a\\a&0&b&0\\0&a&0&b\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\beta\\0&0&-\beta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

Proposition 1.3¹.1. *Any solution of the Einstein–Maxwell equation on the pair* 1.3¹.1 *has the form:*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\beta&\alpha\\0&0&\alpha&\gamma\\-\beta&-\alpha&0&0\\-\alpha&-\gamma&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{3c}{8a^2}+\frac{\beta\gamma-\alpha^2}{a^2},$$

where $\alpha^2+\gamma^2\neq 0,\ bc=d^2,\ 2\alpha d=\gamma b+\beta c$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\beta&0\\0&0&0&0\\-\beta&0&0&0\\0&0&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{3c}{8a^2},$$

where $8\beta^2c=15(bc-d^2)$.

Proposition 1.3¹.2. *The Einstein–Maxwell equation has no solutions on the pair* 1.3¹.2.

Proposition 1.3¹.3. *The Einstein–Maxwell equation has no solutions on the pair* 1.3¹.3.

Proposition 1.3¹.4. *The Einstein–Maxwell equation has no solutions on the pair* 1.3¹.4.

Proposition 1.3¹.5. *Any solution of the Einstein–Maxwell equation on the pair* 1.3¹.5 *has the form:*

$\lambda=0,\ \mu=2$

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\beta&0\\0&0&0&\beta\\-\beta&0&0&-\frac{\beta(b+c)}{2a}\\0&-\beta&\frac{\beta(b+c)}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{\beta^2}{a^2},$$

where $\beta\neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

$\lambda = 0,\ \mu = -2$

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -3\beta \\ -\beta & 0 & 0 & \frac{2a}{3\beta} + \frac{\beta(3b-c)}{2a} \\ 0 & 3\beta & -\frac{2a}{3\beta} - \frac{\beta(3b-c)}{2a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{3\beta^2}{a^2},$$

where $\beta \neq 0$.

For other values of λ, μ *the Einstein–Maxwell equation has no solutions.*

Proposition 1.3¹.6. *The Einstein–Maxwell equation has no solutions on the pair* 1.3¹.6.

Proposition 1.3¹.7. *The Einstein–Maxwell equation has no solutions on the pair* 1.3¹.7.

Proposition 1.3¹.8. *Any solution of the Einstein–Maxwell equation on the pair* 1.3¹.8 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & -\frac{a}{4\gamma} - \frac{\gamma b}{2a} \\ 0 & -\gamma & \frac{a}{4\gamma} + \frac{\gamma b}{2a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\gamma \neq 0$.

Proposition 1.3¹.9. *Any solution of the Einstein–Maxwell equation on the pair* 1.3¹.9 *has the form:*

$\lambda = 1$

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \\ -\beta & 0 & 0 & -\frac{\gamma b + \beta c}{2a} \\ 0 & -\gamma & \frac{\gamma b + \beta c}{2a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta\gamma}{a^2},$$

where $\beta^2 + \gamma^2 \neq 0$, *or*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & -\delta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

$\lambda \neq 1$

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & -\frac{(1-\lambda)^2 a}{4\gamma} - \frac{\gamma b}{2a} \\ 0 & -\gamma & \frac{(1-\lambda)^2 a}{4\gamma} + \frac{\gamma b}{2a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\gamma \neq 0$.

Proposition 1.3¹.10. *Any solution of the Einstein–Maxwell equation on the pair* 1.3¹.10 *has the form:*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&\gamma\\0&0&0&-\frac{a}{4\gamma}-\frac{\gamma b}{2a}\\0&-\gamma&\frac{a}{4\gamma}+\frac{\gamma b}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $\gamma\neq 0$.

Proposition 1.3¹.11. *Any solution of the Einstein–Maxwell equation on the pair* 1.3¹.11 *has the form:*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&\gamma\\0&0&0&-\frac{a}{\gamma}-\frac{\gamma b}{2a}\\0&-\gamma&\frac{a}{\gamma}+\frac{\gamma b}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $\gamma\neq 0$.

Proposition 1.3¹.12. *Any solution of the Einstein–Maxwell equation on the pair* 1.3¹.12 *has the form:*

$\lambda=1,\ \mu=2$

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\beta&\alpha\\0&0&\alpha&\gamma\\-\beta&-\alpha&0&\frac{2\alpha d-\gamma b-\beta c}{2a}\\-\alpha&-\gamma&-\frac{2\alpha d-\gamma b-\beta c}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{\beta\gamma-\alpha^2}{a^2},$$

where $\alpha^2+\beta^2+\gamma^2\neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

$\lambda=3,\ \mu=2$

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\beta&0\\0&0&0&\gamma\\-\beta&0&0&-\frac{\gamma b+\beta c}{2a}\\0&-\gamma&\frac{\gamma b+\beta c}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{\beta\gamma}{a^2},$$

where $\beta^2+\gamma^2\neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

$\mu = \lambda + 1,\ \lambda \neq 1$

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha & \gamma \\ 0 & -\alpha & 0 & \frac{2\alpha d - \gamma b}{2a} \\ -\alpha & -\gamma & -\frac{2\alpha d-\gamma b}{2a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\alpha^2}{a^2},$$

where $\alpha^2 + \gamma^2 \neq 0$, *or*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & -\delta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

$\mu = \lambda - 1,\ \lambda \neq 3$

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & -\frac{\gamma b}{2a} \\ 0 & -\gamma & \frac{\gamma b}{2a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\gamma \neq 0$, *or*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & -\delta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

$\mu \notin \{\lambda - 1, \lambda + 1\}$

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \ \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & \frac{(1-(\lambda-\mu)^2)a}{4\gamma} - \frac{\gamma b}{2a} \\ 0 & -\gamma & -\frac{(1-(\lambda-\mu)^2)a}{4\gamma} + \frac{\gamma b}{2a} & 0 \end{pmatrix}, \ \boldsymbol{\lambda} = 0,$$

where $\gamma \neq 0$.

Proposition 1.3¹.13. *Any solution of the Einstein–Maxwell equation on the pair* 1.3¹.13 *has the form:*

$\lambda = -\frac{1}{2}$

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha & \gamma \\ 0 & -\alpha & 0 & \frac{2\alpha d - \gamma b}{2a} \\ -\alpha & -\gamma & -\frac{2\alpha d-\gamma b}{2a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\alpha^2}{a^2},$$

where $\alpha^2 + \gamma^2 \neq 0$, *or*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & -\delta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

$\lambda = \frac{3}{2}$

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & -\frac{\gamma b}{2a} \\ 0 & -\gamma & \frac{\gamma b}{2a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\gamma \neq 0$, *or*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & -\delta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

$\lambda \notin \{-\frac{1}{2}, \frac{3}{2}\}$

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \; \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & \frac{(1-(\lambda-\frac{1}{2})^2)a}{4\gamma} - \frac{\gamma b}{2a} \\ 0 & -\gamma & -\frac{(1-(\lambda-\frac{1}{2})^2)a}{4\gamma} + \frac{\gamma b}{2a} & 0 \end{pmatrix}, \; \boldsymbol{\lambda} = 0,$$

where $\gamma \neq 0$.

Proposition 1.3[1].14. *Any solution of the Einstein–Maxwell equation on the pair* 1.3[1].14 *has the form:*

$\lambda = 0$

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha & \gamma \\ 0 & -\alpha & 0 & \frac{2\alpha d - \gamma b}{2a} \\ -\alpha & -\gamma & -\frac{2\alpha d-\gamma b}{2a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\alpha^2}{a^2},$$

where $\alpha^2 + \gamma^2 \neq 0$, *or*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & -\delta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

$\lambda = 1$

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & -\frac{\gamma b}{2a} \\ 0 & -\gamma & \frac{\gamma b}{2a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\gamma \neq 0$, *or*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & d \\ a & 0 & d & c \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & -\delta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

$\lambda \notin \{0,1\}$

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&\gamma\\0&0&0&\frac{(1-(2\lambda-1)^2)a}{4\gamma}-\frac{\gamma b}{2a}\\0&-\gamma&-\frac{(1-(2\lambda-1)^2)a}{4\gamma}+\frac{\gamma b}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $\gamma \neq 0$.

Proposition $1.3^1.15$. *The Einstein–Maxwell equation has no solutions on the pair* $1.3^1.15$.

Proposition $1.3^1.16$. *The Einstein–Maxwell equation has no solutions on the pair* $1.3^1.16$.

Proposition $1.3^1.17$. *Any solution of the Einstein–Maxwell equation on the pair* $1.3^1.17$ *has the form:*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&\alpha\\0&0&\alpha&\gamma\\0&-\alpha&0&\frac{2\alpha d-\gamma b}{2a}\\-\alpha&-\gamma&-\frac{2\alpha d-\gamma b}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\alpha^2}{a^2},$$

where $\alpha^2+\gamma^2 \neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

Proposition $1.3^1.18$. *Any solution of the Einstein–Maxwell equation on the pair* $1.3^1.18$ *has the form:*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&\alpha\\0&0&\alpha&\gamma\\0&-\alpha&0&\frac{2\alpha d-\gamma b}{2a}\\-\alpha&-\gamma&-\frac{2\alpha d-\gamma b}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\alpha^2}{a^2},$$

where $\alpha^2+\gamma^2 \neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

Proposition $1.3^1.19$. *Any solution of the Einstein–Maxwell equation on the pair* $1.3^1.19$ *has the form:*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&\gamma\\0&0&0&\frac{a}{4\gamma}-\frac{\gamma b}{2a}\\0&-\gamma&-\frac{a}{4\gamma}+\frac{\gamma b}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $\gamma \neq 0$.

Proposition 1.3[1].20. *Any solution of the Einstein–Maxwell equation on the pair* 1.3[1].20 *has the form:*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&\gamma\\0&0&0&-\frac{a}{4\gamma}-\frac{\gamma b}{2a}\\0&-\gamma&\frac{a}{4\gamma}+\frac{\gamma b}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $\gamma\neq 0$.

Proposition 1.3[1].21. *Any solution of the Einstein–Maxwell equation on the pair* 1.3[1].21 *has the form:*

$\lambda=0$

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&\gamma\\0&0&0&-\frac{\gamma b}{2a}\\0&-\gamma&\frac{\gamma b}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $\gamma\neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

$\lambda=2$

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&\alpha\\0&0&\alpha&\gamma\\0&-\alpha&0&\frac{2\alpha d-\gamma b}{2a}\\-\alpha&-\gamma&-\frac{2\alpha d-\gamma b}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\alpha^2}{a^2},$$

where $\alpha^2+\gamma^2\neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

$\lambda\notin\{0,2\}$

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\ \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&\gamma\\0&0&0&\frac{(2\lambda-\lambda^2)a}{4\gamma}-\frac{\gamma b}{2a}\\0&-\gamma&-\frac{(2\lambda-\lambda^2)a}{4\gamma}+\frac{\gamma b}{2a}&0\end{pmatrix},\ \boldsymbol{\lambda}=0,$$

where $\gamma\neq 0$.

Proposition 1.3[1].22. *Any solution of the Einstein–Maxwell equation on the pair* 1.3[1].22 *has the form:*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&\gamma\\0&0&0&\frac{3a}{16\gamma}-\frac{\gamma b}{2a}\\0&-\gamma&-\frac{3a}{16\gamma}+\frac{\gamma b}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $\gamma\neq 0$.

Proposition 1.3[1].23. *Any solution of the Einstein–Maxwell equation on the pair* 1.3[1].23 *has the form:*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&\gamma\\0&0&0&-\frac{\gamma b}{2a}\\0&-\gamma&\frac{\gamma b}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $\gamma\neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

Proposition 1.3[1].24. *Any solution of the Einstein–Maxwell equation on the pair* 1.3[1].24 *has the form:*

$\lambda=0$

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&2\gamma&0\\0&0&0&\gamma\\-2\gamma&0&0&-\frac{\gamma(b+2c)}{2a}-\frac{a}{2\gamma}\\0&-\gamma&\frac{\gamma(b+2c)}{2a}+\frac{a}{2\gamma}&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{2\gamma^2}{a^2},$$

where $\gamma\neq 0$.

$\lambda=2$

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&-2\gamma&\alpha\\0&0&\alpha&\gamma\\2\gamma&-\alpha&0&\frac{\alpha d}{a}+\frac{\gamma(2c-b)}{2a}\\-\alpha&-\gamma&-\frac{\alpha d}{a}-\frac{\gamma(2c-b)}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\alpha^2+2\gamma^2}{a^2},$$

where $\alpha^2+\gamma^2\neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

For other values of λ *the Einstein–Maxwell equations has no solutions.*

Proposition $1.3^1.25$. *Any solution of the Einstein–Maxwell equation on the pair* $1.3^1.25$ *has the form:*

$\lambda = 0$

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&-2\gamma&0\\0&0&0&\gamma\\2\gamma&0&0&\frac{\gamma(2c-b)}{2a}+\frac{a}{2\gamma}\\0&-\gamma&-\frac{\gamma(2c-b)}{2a}-\frac{a}{2\gamma}&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{2\gamma^2}{a^2},$$

where $\gamma \neq 0$.

$\lambda = 2$

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&2\gamma&\alpha\\0&0&\alpha&\gamma\\-2\gamma&-\alpha&0&\frac{\alpha d}{a}-\frac{\gamma(2c+b)}{2a}\\-\alpha&-\gamma&-\frac{\alpha d}{a}+\frac{\gamma(2c+b)}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{2\gamma^2-\alpha^2}{a^2},$$

where $\alpha^2+\gamma^2 \neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

For other values of λ *the Einstein–Maxwell equations has no solutions.*

Proposition $1.3^1.26$. *The Einstein–Maxwell equation has no solutions on the pair* $1.3^1.26$.

Proposition $1.3^1.27$. *The Einstein–Maxwell equation has no solutions on the pair* $1.3^1.27$.

Proposition $1.3^1.28$. *The Einstein–Maxwell equation has no solutions on the pair* $1.3^1.28$.

Proposition $1.3^1.29$. *The Einstein–Maxwell equation has no solutions on the pair* $1.3^1.29$.

Proposition $1.3^1.30$. *Any solution of the Einstein–Maxwell equation on the pair* $1.3^1.30$ *has the form:*

$\lambda = 1,\ \mu = 1$

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&\alpha\\0&0&\alpha&0\\0&-\alpha&0&\frac{\alpha d}{a}\\-\alpha&0&-\frac{\alpha d}{a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\alpha^2}{a^2},$$

where $\alpha \neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

For other values of λ, μ *the Einstein–Maxwell equation has no solutions.*

Proposition 1.3¹.31. *Any solution of the Einstein–Maxwell equation on the pair* 1.3¹.31 *has the form:*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\beta&\alpha\\0&0&\alpha&\gamma\\-\beta&-\alpha&0&\frac{2\alpha d-\gamma b-\beta c}{2a}\\-\alpha&-\gamma&-\frac{2\alpha d-\gamma b-\beta c}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{\beta\gamma-\alpha^2}{a^2},$$

where $\alpha^2+\beta^2+\gamma^2\neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

Proposition 1.3¹.32. *Any solution of the Einstein–Maxwell equation on the pair* 1.3¹.32 *has the form:*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\beta&\alpha\\0&0&\alpha&\gamma\\-\beta&-\alpha&0&\frac{2\alpha d-\gamma b-\beta c}{2a}\\-\alpha&-\gamma&-\frac{2\alpha d-\gamma b-\beta c}{2a}&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{\beta\gamma-\alpha^2}{a^2},$$

where $\alpha^2+\beta^2+\gamma^2\neq 0$, *or*

$$B=\begin{pmatrix}0&0&0&a\\0&0&-a&0\\0&-a&b&d\\a&0&d&c\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\delta\\0&0&-\delta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

Proposition 1.4¹.1. *The Einstein–Maxwell equation has no solutions on the pair* 1.4¹.1.

Proposition 1.4¹.2. *Any solution of the Einstein–Maxwell equation on the pair* 1.4¹.2 *has the form:*

$p=\frac{3}{2}$

$$B=\begin{pmatrix}0&0&-a&0\\0&a&0&0\\-a&0&b&c\\0&0&c&d\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\alpha&0\\0&-\alpha&0&\beta\\0&0&-\beta&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{3}{4d},$$

where $\frac{b}{2d}=\frac{\alpha^2}{a}+\frac{\beta^2}{d}$.

$p=\frac{5}{3}$

$$B=\begin{pmatrix}0&0&-a&0\\0&a&0&0\\-a&0&b&c\\0&0&c&d\end{pmatrix},\quad \Omega=0,\quad \boldsymbol{\lambda}=-\frac{4}{3d}.$$

$p\notin\{\frac{3}{2},\frac{5}{3}\}$

$$B=\begin{pmatrix}0&0&-a&0\\0&a&0&0\\-a&0&0&c\\0&0&c&d\end{pmatrix},\quad \Omega=0,\quad \boldsymbol{\lambda}=-\frac{3(p-1)^2}{d}.$$

Proposition 1.4[1].3. *Any solution of the Einstein–Maxwell equation on the pair* 1.4[1].3 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & d & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{3}{d}.$$

Proposition 1.4[1].4. *Any solution of the Einstein–Maxwell equation on the pair* 1.4[1].4 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & -d & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{3}{d}.$$

Proposition 1.4[1].5. *The Einstein–Maxwell equation has no solutions on the pair* 1.4[1].5.

Proposition 1.4[1].6. *The Einstein–Maxwell equation has no solutions on the pair* 1.4[1].6.

Proposition 1.4[1].7. *The Einstein–Maxwell equation has no solutions on the pair* 1.4[1].7.

Proposition 1.4[1].8. *Any solution of the Einstein–Maxwell equation on the pair* 1.4[1].8 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{3}{d}.$$

Proposition 1.4[1].9. *Any solution of the Einstein–Maxwell equation on the pair* 1.4[1].9 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $p^2 + p + r + \frac{d}{2a} = \frac{\alpha^2}{a} + \frac{\beta^2}{d}$.

Proposition 1.4[1].10. *Any solution of the Einstein–Maxwell equation on the pair* 1.4[1].10 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $p^2 + p + r = \frac{\alpha^2}{a} + \frac{\beta^2}{d}$.

Proposition 1.4[1].11. *Any solution of the Einstein–Maxwell equation on the pair* 1.4[1].11 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $r + \frac{d}{2a} = \frac{\alpha^2}{a} + \frac{\beta^2}{d}$.

Proposition 1.4^1.12. *Any solution of the Einstein–Maxwell equation on the pair* 1.4^1.12 *has the form:*

$$B=\begin{pmatrix}0&0&-a&0\\0&a&0&0\\-a&0&b&c\\0&0&c&d\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\alpha&0\\0&-\alpha&0&\beta\\0&0&-\beta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $r=\frac{\alpha^2}{a}+\frac{\beta^2}{d}$.

Proposition 1.4^1.13. *Any solution of the Einstein–Maxwell equation on the pair* 1.4^1.13 *has the form:*

$$B=\begin{pmatrix}0&0&-a&0\\0&a&0&0\\-a&0&b&c\\0&0&c&d\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\alpha&0\\0&-\alpha&0&\beta\\0&0&-\beta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $r+1+\frac{d}{2a}=\frac{\alpha^2}{a}+\frac{\beta^2}{d}$.

Proposition 1.4^1.14. *Any solution of the Einstein–Maxwell equation on the pair* 1.4^1.14 *has the form:*

$$B=\begin{pmatrix}0&0&-a&0\\0&a&0&0\\-a&0&b&c\\0&0&c&d\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\alpha&0\\0&-\alpha&0&\beta\\0&0&-\beta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $r+1=\frac{\alpha^2}{a}+\frac{\beta^2}{d}$.

Proposition 1.4^1.15. *Any solution of the Einstein–Maxwell equation on the pair* 1.4^1.15 *has the form:*

$$B=\begin{pmatrix}0&0&-a&0\\0&a&0&0\\-a&0&b&c\\0&0&c&d\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\alpha&0\\0&-\alpha&0&\beta\\0&0&-\beta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $1+\frac{d}{2a}=\frac{\alpha^2}{a}+\frac{\beta^2}{d}$.

Proposition 1.4^1.16. *Any solution of the Einstein–Maxwell equation on the pair* 1.4^1.16 *has the form:*

$$B=\begin{pmatrix}0&0&-a&0\\0&a&0&0\\-a&0&b&c\\0&0&c&d\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\alpha&0\\0&-\alpha&0&\beta\\0&0&-\beta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $\frac{d}{2a}-1=\frac{\alpha^2}{a}+\frac{\beta^2}{d}$.

Proposition 1.4[1].17. *Any solution of the Einstein–Maxwell equation on the pair* 1.4[1].17 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\frac{d}{2a} = \frac{\alpha^2}{a} + \frac{\beta^2}{d}$.

Proposition 1.4[1].18. *Any solution of the Einstein–Maxwell equation on the pair* 1.4[1].18 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $1 + \frac{d}{2a} = \frac{\alpha^2}{a} + \frac{\beta^2}{d}$.

Proposition 1.4[1].19. *Any solution of the Einstein–Maxwell equation on the pair* 1.4[1].19 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\frac{d}{2a} - 1 = \frac{\alpha^2}{a} + \frac{\beta^2}{d}$.

Proposition 1.4[1].20. *Any solution of the Einstein–Maxwell equation on the pair* 1.4[1].20 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\frac{d}{2a} = \frac{\alpha^2}{a} + \frac{\beta^2}{d}$.

Proposition 1.4[1].21. *Any solution of the Einstein–Maxwell equation on the pair* 1.4[1].21 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\frac{\alpha^2}{a} + \frac{\beta^2}{d} = 1$.

Proposition 1.4^1.22. *Any solution of the Einstein–Maxwell equation on the pair* 1.4^1.22 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\frac{\alpha^2}{a} + \frac{\beta^2}{d} = -1$.

Proposition 1.4^1.23. *Any solution of the Einstein–Maxwell equation on the pair* 1.4^1.23 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\frac{\alpha^2}{a} + \frac{\beta^2}{d} = 0$.

Proposition 1.4^1.24. *Any solution of the Einstein–Maxwell equation on the pair* 1.4^1.24 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\frac{\alpha^2}{a} + \frac{\beta^2}{d} = 1$.

Proposition 1.4^1.25. *Any solution of the Einstein–Maxwell equation on the pair* 1.4^1.25 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\frac{\alpha^2}{a} + \frac{\beta^2}{d} = -1$.

Proposition 1.4^1.26. *Any solution of the Einstein–Maxwell equation on the pair* 1.4^1.26 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\frac{\alpha^2}{a} + \frac{\beta^2}{d} = 0$.

Proposition 2.1[1].1. *Any solution of the Einstein–Maxwell equation on the pair* 2.1[1].1 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\beta^2 + b}{b^2},$$

where $\alpha^2 = \frac{b+\beta^2}{b^2}a^2 - a$.

Proposition 2.1[1].2. *Any solution of the Einstein–Maxwell equation on the pair* 2.1[1].2 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\beta^2}{b^2},$$

where $\alpha^2 = \frac{\beta^2}{b^2}a^2 - a$.

Proposition 2.1[1].3. *Any solution of the Einstein–Maxwell equation on the pair* 2.1[1].3 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\beta^2}{b^2},$$

where $\alpha^2 = \frac{\beta^2}{b^2}a^2$.

Proposition 2.1[2].1. *Any solution of the Einstein–Maxwell equation on the pair* 2.1[2].1 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\alpha^2 + a}{a^2},$$

where $\beta^2 = -\frac{\alpha^2+a}{a^2}b^2 - b$.

Proposition 2.1[2].2. *Any solution of the Einstein–Maxwell equation on the pair* 2.1[2].2 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\alpha^2 + a}{a^2},$$

where $\beta^2 = -\frac{\alpha^2+a}{a^2}b^2 + b$.

Proposition $2.1^2.3$. *Any solution of the Einstein–Maxwell equation on the pair* $2.1^2.3$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\alpha^2 + a}{a^2},$$

where $\beta^2 = -\frac{\alpha^2+a}{a^2}b^2$.

Proposition $2.1^2.4$. *Any solution of the Einstein–Maxwell equation on the pair* $2.1^2.4$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\alpha^2}{a^2},$$

where $\beta^2 = -\frac{\alpha^2}{a^2}b^2 - b$.

Proposition $2.1^2.5$. *Any solution of the Einstein–Maxwell equation on the pair* $2.1^2.5$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\alpha^2}{a^2},$$

where $\beta^2 = -\frac{\alpha^2}{a^2}b^2 + b$.

Proposition $2.1^2.6$. *Any solution of the Einstein–Maxwell equation on the pair* $2.1^2.6$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition $2.1^3.1$. *Any solution of the Einstein–Maxwell equation on the pair* $2.1^3.1$ *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2 + b}{b^2},$$

where $\alpha^2 = \frac{b+\beta^2}{b^2}a^2 - a$.

Proposition $2.1^3.2$. *Any solution of the Einstein–Maxwell equation on the pair* $2.1^3.2$ *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2 - b}{b^2},$$

where $\alpha^2 = \frac{\beta^2-b}{b^2}a^2 - a$.

Proposition $2.1^3.3$. *Any solution of the Einstein–Maxwell equation on the pair* $2.1^3.3$ *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2 - b}{b^2},$$

where $\alpha^2 = \frac{\beta^2 - b}{b^2} a^2 + a$.

Proposition $2.1^3.4$. *Any solution of the Einstein–Maxwell equation on the pair* $2.1^3.4$ *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2}{b^2},$$

where $\alpha^2 = \frac{\beta^2}{b^2} a^2 - a$.

Proposition $2.1^3.5$. *Any solution of the Einstein–Maxwell equation on the pair* $2.1^3.5$ *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2}{b^2},$$

where $\alpha^2 = \frac{\beta^2}{b^2} a^2 + a$.

Proposition $2.1^3.6$. *Any solution of the Einstein–Maxwell equation on the pair* $2.1^3.6$ *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta^2}{b^2},$$

where $\alpha^2 = \frac{\beta^2}{b^2} a^2$.

Proposition $2.1^4.1$. *Any solution of the Einstein–Maxwell equation on the pair* $2.1^4.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & b & -a \\ a & b & 0 & 0 \\ b & -a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & \beta \\ 0 & 0 & \beta & -\alpha \\ -\alpha & -\beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{2a + \alpha^2 - \beta^2}{b^2 - a^2},$$

where $a^2 \neq b^2$, $\alpha\beta = \frac{\beta^2 - \alpha^2 - 2a}{b^2 - a^2} ab - b$, *or*

$$B = \begin{pmatrix} 0 & 0 & a & \varepsilon a \\ 0 & 0 & \varepsilon a & -a \\ a & \varepsilon a & 0 & 0 \\ \varepsilon a & -a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & \beta \\ 0 & 0 & \beta & -\alpha \\ -\alpha & -\beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 2\frac{\varepsilon(\alpha^2 - \beta^2) - 2\alpha\beta}{\varepsilon(\beta^2 - \alpha^2)^2},$$

where $\alpha^2 \neq \beta^2$, $a = \frac{\beta^2 - \alpha^2}{2}$, $\varepsilon \in \{-1, 1\}$.

Proposition 2.1^4.2. *Any solution of the Einstein–Maxwell equation on the pair* 2.1^4.2 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & b & -a \\ a & b & 0 & 0 \\ b & -a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & \beta \\ 0 & 0 & \beta & -\alpha \\ -\alpha & -\beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\alpha^2 - \beta^2}{b^2 - a^2},$$

where $a^2 \neq b^2$, $\alpha\beta = \frac{\beta^2 - \alpha^2}{b^2 - a^2} ab$, *or*

$$B = \begin{pmatrix} 0 & 0 & a & \varepsilon a \\ 0 & 0 & \varepsilon a & -a \\ a & \varepsilon a & 0 & 0 \\ \varepsilon a & -a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & \delta\alpha \\ 0 & 0 & \delta\alpha & -\alpha \\ -\alpha & -\delta\alpha & 0 & 0 \\ -\delta\alpha & \alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\delta\alpha^2}{\varepsilon a^2},$$

where $\varepsilon \in \{-1, 1\}$, $\delta \in \{-1, 1\}$.

Proposition 2.2^1.1. *Any solution of the Einstein–Maxwell equation on the pair* 2.2^1.1 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{3b - 2\alpha^2}{2a^2}.$$

Proposition 2.2^1.2. *Any solution of the Einstein–Maxwell equation on the pair* 2.2^1.2 *has the form:*

$p = 2$

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\alpha^2}{a^2}.$$

$p = -2$

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

For other values of p *the Einstein–Maxwell equation has no solutions.*

Proposition 2.2^1.3. *The Einstein–Maxwell equation has no solutions on the pair* 2.2^1.3.

Proposition 2.2^1.4. *Any solution of the Einstein–Maxwell equation on the pair* 2.2^1.4 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & b & a \\ a & b & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & \frac{\alpha^2 b - a^2}{\alpha a} & \alpha \\ -\alpha & -\frac{\alpha^2 b - a^2}{\alpha a} & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\alpha^2}{a^2},$$

where $\alpha \neq 0$.

Proposition 2.2[1].5. *Any solution of the Einstein–Maxwell equation on the pair* 2.2[1].5 *has the form:*

$$B=\begin{pmatrix}0&0&a&0\\0&0&b&a\\a&b&0&0\\0&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\alpha&0\\0&0&\frac{\alpha b}{a}&\alpha\\-\alpha&-\frac{\alpha b}{a}&0&0\\0&-\alpha&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\alpha^2}{a^2},$$

where $\alpha\neq 0$, *or*

$$B=\begin{pmatrix}0&0&a&0\\0&0&b&a\\a&b&0&0\\0&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\beta&0\\0&-\beta&0&0\\0&0&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

Proposition 2.2[1].6. *Any solution of the Einstein–Maxwell equation on the pair* 2.2[1].6 *has the form:*

$$B=\begin{pmatrix}0&0&a&0\\0&0&0&a\\a&0&0&0\\0&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\alpha&0\\0&0&0&\alpha\\-\alpha&0&0&0\\0&-\alpha&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\alpha^2}{a^2}.$$

Proposition 2.2[1].7. *Any solution of the Einstein–Maxwell equation on the pair* 2.2[1].7 *has the form:*

$\lambda=0$

$$B=\begin{pmatrix}0&0&a&0\\0&b&0&a\\a&0&0&0\\0&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\alpha&0\\0&0&0&\alpha\\-\alpha&0&0&0\\0&-\alpha&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\alpha^2}{a^2}.$$

$\lambda=1$

$$B=\begin{pmatrix}0&0&a&0\\0&0&b&a\\a&b&0&0\\0&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\alpha&0\\0&0&\frac{\alpha b}{a}&\alpha\\-\alpha&-\frac{\alpha b}{a}&0&0\\0&-\alpha&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\alpha^2}{a^2},$$

where $\alpha\neq 0$, *or*

$$B=\begin{pmatrix}0&0&a&0\\0&0&b&a\\a&b&0&0\\0&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\beta&0\\0&-\beta&0&0\\0&0&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=0.$$

$\lambda=-1$

$$B=\begin{pmatrix}0&0&a&0\\0&0&0&a\\a&0&0&0\\0&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&\beta&\alpha&0\\-\beta&0&0&\alpha\\-\alpha&0&0&\gamma\\0&-\alpha&-\gamma&0\end{pmatrix},\quad \boldsymbol{\lambda}=\frac{\beta\gamma-\alpha^2}{a^2}.$$

$\lambda\in]-1,1[\setminus\{0\}$

$$B=\begin{pmatrix}0&0&a&0\\0&0&0&a\\a&0&0&0\\0&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&\alpha&0\\0&0&0&\alpha\\-\alpha&0&0&0\\0&-\alpha&0&0\end{pmatrix},\quad \boldsymbol{\lambda}=-\frac{\alpha^2}{a^2}.$$

Proposition 2.2^2.1. *Any solution of the Einstein–Maxwell equation on the pair* 2.2^2.1 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & 0 \\ a & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & -\frac{\alpha^2 b + a^2}{\alpha a} \\ 0 & -\alpha & \frac{\alpha^2 b + a^2}{\alpha a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\alpha^2}{a^2},$$

where $\alpha \neq 0$.

Proposition 2.2^2.2. *Any solution of the Einstein–Maxwell equation on the pair* 2.2^2.2 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & 0 \\ a & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & \frac{a^2 - \alpha^2 b}{\alpha a} \\ 0 & -\alpha & -\frac{a^2 - \alpha^2 b}{\alpha a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\alpha^2}{a^2},$$

where $\alpha \neq 0$.

Proposition 2.2^2.3. *Any solution of the Einstein–Maxwell equation on the pair* 2.2^2.3 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & 0 \\ a & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & -\frac{\alpha b}{a} \\ 0 & -\alpha & \frac{\alpha b}{a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\alpha^2}{a^2},$$

where $\alpha \neq 0$, *or*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & 0 \\ a & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

Proposition 2.2^2.4. *Any solution of the Einstein–Maxwell equation on the pair* 2.2^2.4 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & 0 \\ a & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & -\frac{\alpha b}{a} \\ 0 & -\alpha & \frac{\alpha b}{a} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\alpha^2}{a^2},$$

where $\alpha \neq 0$, *or*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & 0 \\ a & 0 & 0 & b \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

Proposition 2.2^3.1. *Any solution of the Einstein–Maxwell equation on the pair* 2.2^3.1 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\alpha^2}{a^2}.$$

Proposition 2.3^1.1. *Any solution of the Einstein–Maxwell equation on the pair* 2.3^1.1 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

Proposition 2.4^1.1. *The Einstein–Maxwell equation has no solutions on the pair* 2.4^1.1.

Proposition 2.4^1.2. *Any solution of the Einstein–Maxwell equation on the pair* 2.4^1.2 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{3}{b}.$$

Proposition 2.4^1.3. *Any solution of the Einstein–Maxwell equation on the pair* 2.4^1.3 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition 2.5^1.1. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^1.1 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{6}{a}.$$

Proposition 2.5^1.2. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^1.2 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

Proposition $2.5^1.3$. *Any solution of the Einstein–Maxwell equation on the pair* $2.5^1.3$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $h - \frac{h^2}{2} + 2g = \frac{2\alpha\beta}{a}$.

Proposition $2.5^1.4$. *Any solution of the Einstein–Maxwell equation on the pair* $2.5^1.4$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $h - \frac{h^2}{2} + 2g = \frac{2\alpha\beta}{a}$.

Proposition $2.5^1.5$. *Any solution of the Einstein–Maxwell equation on the pair* $2.5^1.5$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $2g - \frac{1}{2} = \frac{2\alpha\beta}{a}$.

Proposition $2.5^1.6$. *Any solution of the Einstein–Maxwell equation on the pair* $2.5^1.6$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $2g - \frac{1}{2} = \frac{2\alpha\beta}{a}$.

Proposition $2.5^1.7$. *Any solution of the Einstein–Maxwell equation on the pair* $2.5^1.7$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \frac{a}{\alpha} \\ 0 & 0 & -\frac{a}{\alpha} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\alpha \neq 0$.

Proposition 2.5^1.8. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^1.8 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & -\frac{a}{\alpha} \\ 0 & 0 & \frac{a}{\alpha} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\alpha \neq 0$.

Proposition 2.5^1.9. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^1.9 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \frac{a}{\alpha} \\ 0 & 0 & -\frac{a}{\alpha} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\alpha \neq 0$.

Proposition 2.5^1.10. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^1.10 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & -\frac{a}{\alpha} \\ 0 & 0 & \frac{a}{\alpha} & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\alpha \neq 0$.

Proposition 2.5^1.11. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^1.11 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\alpha\beta = 0$.

Proposition 2.5^1.12. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^1.12 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $\alpha\beta = 0$.

Proposition 2.5^1.13. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^1.13 *has the form:*

$$B=\begin{pmatrix}0&0&a&0\\0&0&0&a\\a&0&b&0\\0&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\alpha&0\\0&-\alpha&0&\beta\\0&0&-\beta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $\alpha\beta=0$.

Proposition 2.5^1.14. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^1.14 *has the form:*

$$B=\begin{pmatrix}0&0&a&0\\0&0&0&a\\a&0&b&0\\0&a&0&0\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\alpha&0\\0&-\alpha&0&\beta\\0&0&-\beta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $\alpha\beta=0$.

Proposition 2.5^2.1. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^2.1 *has the form:*

$$B=\begin{pmatrix}0&0&a&0\\0&a&0&0\\a&0&0&0\\0&0&0&a\end{pmatrix},\quad \Omega=0,\quad \boldsymbol{\lambda}=-\frac{3}{a}.$$

Proposition 2.5^2.2. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^2.2 *has the form:*

$$B=\begin{pmatrix}0&0&a&0\\0&a&0&0\\a&0&b&0\\0&0&0&a\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\alpha&0\\0&-\alpha&0&\beta\\0&0&-\beta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $2r^2+2p=-\frac{\alpha^2+\beta^2}{a}$.

Proposition 2.5^2.3. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^2.3 *has the form:*

$$B=\begin{pmatrix}0&0&a&0\\0&a&0&0\\a&0&b&0\\0&0&0&a\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\alpha&0\\0&-\alpha&0&\beta\\0&0&-\beta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $2r-\frac{1}{2}=\frac{\alpha^2+\beta^2}{a}$.

Proposition 2.5^2.4. *Any solution of the Einstein–Maxwell equation on the pair* 2.5^2.4 *has the form:*

$$B=\begin{pmatrix}0&0&a&0\\0&a&0&0\\a&0&b&0\\0&0&0&a\end{pmatrix},\quad \Omega=\begin{pmatrix}0&0&0&0\\0&0&\alpha&0\\0&-\alpha&0&\beta\\0&0&-\beta&0\end{pmatrix},\quad \boldsymbol{\lambda}=0,$$

where $2a=-(\alpha^2+\beta^2),\ \alpha^2+\beta^2\neq 0$.

Proposition $2.5^2.5$. *Any solution of the Einstein–Maxwell equation on the pair* $2.5^2.5$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0,$$

where $2a = \alpha^2 + \beta^2$, $\alpha^2 + \beta^2 \neq 0$.

Proposition $2.5^2.6$. *Any solution of the Einstein–Maxwell equation on the pair* $2.5^2.6$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition $2.5^2.7$. *Any solution of the Einstein–Maxwell equation on the pair* $2.5^2.7$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition $3.1^1.1$. *Any solution of the Einstein–Maxwell equation on the pair* $3.1^1.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\alpha^2}{a^2}.$$

Proposition $3.1^2.1$. *Any solution of the Einstein–Maxwell equation on the pair* $3.1^2.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\alpha^2}{a^2}.$$

Proposition $3.2^1.1$. *Any solution of the Einstein–Maxwell equation on the pair* $3.2^1.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{6}{a}.$$

Proposition 3.2^1.2. *Any solution of the Einstein–Maxwell equation on the pair* 3.2^1.2 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition 3.2^1.3. *Any solution of the Einstein–Maxwell equation on the pair* 3.2^1.3 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

Proposition 3.2^1.4. *Any solution of the Einstein–Maxwell equation on the pair* 3.2^1.4 *has the form:*

$\lambda = 1$

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

$\lambda \neq 1$

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition 3.2^2.1. *Any solution of the Einstein–Maxwell equation on the pair* 3.2^2.1 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{3}{a}.$$

Proposition 3.2^2.2. *Any solution of the Einstein–Maxwell equation on the pair* 3.2^2.2 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition 3.3^1.1. *Any solution of the Einstein–Maxwell equation on the pair* 3.3^1.1 *has the form:*

$p = 0$

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

For other values of p the Einstein–Maxwell equation has no solutions.

Proposition $3.3^1.2$. *The Einstein–Maxwell equation has no solutions on the pair* $3.3^1.2$.

Proposition $3.3^1.3$. *The Einstein–Maxwell equation has no solutions on the pair* $3.3^1.3$.

Proposition $3.3^1.4$. *Any solution of the Einstein–Maxwell equation on the pair* $3.3^1.4$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition $3.3^2.1$. *Any solution of the Einstein–Maxwell equation on the pair* $3.3^2.1$ *has the form:*

$p = 0$

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

For other values of p *the Einstein–Maxwell equation has no solutions.*

Proposition $3.3^2.2$. *The Einstein–Maxwell equation has no solutions on the pair* $3.3^2.2$.

Proposition $3.3^2.3$. *The Einstein–Maxwell equation has no solutions on the pair* $3.3^2.3$.

Proposition $3.3^2.4$. *Any solution of the Einstein–Maxwell equation on the pair* $3.3^2.4$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition $3.4^1.1$. *Any solution of the Einstein–Maxwell equation on the pair* $3.4^1.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & \beta & \alpha & 0 \\ -\beta & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & \gamma \\ 0 & -\alpha & -\gamma & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\beta\gamma - \alpha^2}{a^2}.$$

Proposition $3.4^2.1$. *Any solution of the Einstein–Maxwell equation on the pair* $3.4^2.1$ *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & -\gamma & \beta \\ -\beta & \gamma & 0 & -\alpha \\ -\gamma & -\beta & \alpha & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\alpha^2 + \beta^2 + \gamma^2}{a^2}.$$

Proposition $3.5^1.1$. *Any solution of the Einstein–Maxwell equation on the pair* $3.5^1.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & 2a & 0 \\ 0 & a & 0 & 0 \\ 2a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{3}{b}.$$

Proposition $3.5^1.2$. *The Einstein–Maxwell equation has no solutions on the pair* $3.5^1.2$.

Proposition $3.5^1.3$. *The Einstein–Maxwell equation has no solutions on the pair* $3.5^1.3$.

Proposition $3.5^1.4$. *Any solution of the Einstein–Maxwell equation on the pair* $3.5^1.4$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & 2a & 0 \\ 0 & a & 0 & 0 \\ 2a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition $3.5^2.1$. *Any solution of the Einstein–Maxwell equation on the pair* $3.5^2.1$ *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{3}{b}.$$

Proposition $3.5^2.2$. *The Einstein–Maxwell equation has no solutions on the pair* $3.5^2.2$.

Proposition $3.5^2.3$. *The Einstein–Maxwell equation has no solutions on the pair* $3.5^2.3$.

Proposition $3.5^2.4$. *Any solution of the Einstein–Maxwell equation on the pair* $3.5^2.4$ *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition $4.1^1.1$. *Any solution of the Einstein–Maxwell equation on the pair* $4.1^1.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition $4.1^2.1$. *Any solution of the Einstein–Maxwell equation on the pair* $4.1^2.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition $4.2^1.1$. *Any solution of the Einstein–Maxwell equation on the pair* $4.2^1.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{6a + \alpha^2}{a^2}.$$

Proposition 4.2^1.2. *Any solution of the Einstein–Maxwell equation on the pair* 4.2^1.2 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = -\frac{\alpha^2}{a^2}.$$

Proposition 4.2^2.1. *Any solution of the Einstein–Maxwell equation on the pair* 4.2^2.1 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{6a+\alpha^2}{a^2}.$$

Proposition 4.2^2.2. *Any solution of the Einstein–Maxwell equation on the pair* 4.2^2.2 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\alpha^2-6a}{a^2}.$$

Proposition 4.2^2.3. *Any solution of the Einstein–Maxwell equation on the pair* 4.2^2.3 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\alpha^2}{a^2}.$$

Proposition 4.2^3.1. *Any solution of the Einstein–Maxwell equation on the pair* 4.2^3.1 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\alpha^2-6a}{a^2}.$$

Proposition 4.2^3.2. *Any solution of the Einstein–Maxwell equation on the pair* 4.2^3.2 *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{\alpha^2}{a^2}.$$

Proposition $4.3^1.1$. *Any solution of the Einstein–Maxwell equation on the pair* $4.3^1.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

Proposition $4.3^1.2$. *Any solution of the Einstein–Maxwell equation on the pair* $4.3^1.2$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = 0.$$

Proposition $5.1^1.1$. *Any solution of the Einstein–Maxwell equation on the pair* $5.1^1.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition $6.1^1.1$. *Any solution of the Einstein–Maxwell equation on the pair* $6.1^1.1$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{6}{a}.$$

Proposition $6.1^1.2$. *Any solution of the Einstein–Maxwell equation on the pair* $6.1^1.2$ *has the form:*

$$B = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition $6.1^2.1$. *Any solution of the Einstein–Maxwell equation on the pair* $6.1^2.1$ *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{3}{a}.$$

Proposition $6.1^2.2$. *Any solution of the Einstein–Maxwell equation on the pair* $6.1^2.2$ *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = \frac{3}{a}.$$

Proposition 6.1^2.3. *Any solution of the Einstein–Maxwell equation on the pair* 6.1^2.3 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

Proposition 6.1^3.1. *Any solution of the Einstein–Maxwell equation on the pair* 6.1^3.1 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = -\frac{3}{a}.$$

Proposition 6.1^3.2. *Any solution of the Einstein–Maxwell equation on the pair* 6.1^3.2 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = \frac{3}{a}.$$

Proposition 6.1^3.3. *Any solution of the Einstein–Maxwell equation on the pair* 6.1^3.3 *has the form:*

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}, \quad \Omega = 0, \quad \boldsymbol{\lambda} = 0.$$

2. Four-dimensional Einstein–Maxwell homogeneous spaces

A pseudo-Riemannian homogeneous space $(\overline{G}, M)$ is said to be an Einstein–Maxwell homogeneous space if the Einstein–Maxwell equation has a solution on $(\overline{G}, M)$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ corresponding to an Einstein–Maxwell homogeneous space $(\overline{G}, M)$ will be called an Einstein–Maxwell pair. We can divide the four-dimensional Einstein–Maxwell homogeneous spaces (the Einstein–Maxwell pairs of codimension 4) into the Riemannian (with an invariant Riemannian metric), the Lorentzian (with an invariant pseudo-Riemannian metric of signature (3,1)), and the spaces (pairs) of type (2,2) (with an invariant pseudo-Riemannian metric of signature (2,2)). Using the results of Chapter II, 1, we can prove the following Theorems:

Theorem 1. *Any Riemannian Einstein–Maxwell pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 4 is equivalent to one and only one of the following pairs:* $1.1^2.1$, $1.1^2.2$ $(p=1)$, $1.1^2.6$–$1.1^2.12$, $2.1^3.1$–$2.1^3.6$, $3.4^2.1$, $3.5^2.1$, $3.5^2.4$, $4.2^2.1$–$4.2^2.3$, $6.1^2.1$–$6.1^2.3$.

Theorem 2. *Any Lorentzian Einstein–Maxwell pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 4 is equivalent to one and only one of the following pairs:* $1.1^1.2$ $(p=\frac{1}{2})$, $1.1^1.5$–$1.1^1.7$, $1.1^1.10$ $(\lambda=0)$, $1.1^2.2$ $(p=1)$, $1.1^2.6$–$1.1^2.10$, $1.1^2.12$ $(\lambda=0)$, $1.1^3.1$, $1.1^4.1$, $1.4^1.2$–$1.4^1.4$, $1.4^1.8$–$1.4^1.26$, $2.1^2.1$–$2.1^2.6$, $2.4^1.2$, $2.4^1.3$, $2.5^2.1$–$2.5^2.7$, $3.2^2.1$, $3.2^2.2$, $3.3^2.1$ $(p=0)$, $3.3^2.4$, $3.5^1.1$, $3.5^1.4$, $3.5^2.1$, $3.5^2.4$, $4.1^2.1$, $6.1^3.1$–$6.1^3.3$.

Theorem 3. *Any Einstein–Maxwell pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 4 with an invariant pseudo-Riemannian metric of signature (2,2) is equivalent to one and only one of the following pairs:* $1.1^1.1$, $1.1^1.2$ $(p=\frac{1}{2})$, $1.1^1.5$–$1.1^1.10$, $1.1^2.1$, $1.1^2.2$ $(p=1)$, $1.1^2.6$–$1.1^2.12$, $1.1^5.1$, $1.1^6.1$, $1.2^1.1$, $1.2^2.1$, $1.3^1.1$, $1.3^1.5$ $(\lambda=0,\ \mu\in\{-2,2\})$, $1.3^1.8$–$1.3^1.14$, $1.3^1.17$–$1.3^1.23$, $1.3^1.24$ $(\lambda\in\{0,2\})$, $1.3^1.25$ $(\lambda\in\{0,2\})$, $1.3^1.30$ $(\lambda=\mu=1)$, $1.3^1.31$, $1.3^1.32$, $1.4^1.2$–$1.4^1.4$, $1.4^1.8$–$1.4^1.26$, $2.1^1.1$–$2.1^1.3$, $2.1^3.1$–$2.1^3.6$, $2.1^4.1$, $2.1^4.2$, $2.2^1.1$, $2.2^1.2$ $(p\in\{-2,2\})$, $2.2^1.4$–$2.2^1.7$, $2.2^2.1$–$2.2^2.4$, $2.2^3.1$, $2.3^1.1$, $2.4^1.2$, $2.4^1.3$, $2.5^1.1$–$2.5^1.14$, $3.1^1.1$, $3.1^2.1$, $3.2^1.1$–$3.2^1.4$, $3.3^1.1$ $(p=0)$, $3.3^1.4$, $3.4^1.1$, $3.5^1.1$, $3.5^1.4$, $4.1^1.1$, $4.2^1.1$, $4.2^1.2$, $4.2^3.1$, $4.2^3.2$, $4.3^1.1$, $4.3^1.2$, $5.1^1.1$, $6.1^1.1$, $6.1^1.2$.

The solution of the Einstein–Maxwell equation on the pair $m.n^k.i$ of a fixed signature can be found from Proposition $m.n^k.i$. In addition, the metric B satisfies the following complementary conditions:

1. If the signature of B is (3,1) (or (1,3)), then $\det B < 0$.
2. If the signature of B is (4,0) (or (0,4)), then $\det B > 0$ and B is positive (or negative) definite.
3. If the signature of B is (2,2), then $\det B > 0$ and B is neither positive, nor negative definite.

Example. Consider the pair $1.1^2.1$. The invariant pseudo-Riemannian metric on the pair $1.1^2.1$ has the form:

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix}, \qquad \det B = a^2(bd - c^2).$$

It follows that the pair $1.1^2.1$ admits an invariant pseudo-Riemannian metric of arbitrary signature, i.e., the pair $1.1^2.1$ is a Riemannian pair if $bd - c^2 > 0,\ ab > 0$, a Lorentzian pair if $bd - c^2 < 0$, and a pair of type (2,2) if $bd - c^2 > 0,\ ab < 0$.

By Proposition $1.1^2.1$, any solution of the Einstein–Maxwell equation on the pair $1.1^2.1$ has the form:

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix},$$

where $4a^2 = bd - c^2$,

$$\Omega = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -2\alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & 2\alpha & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \frac{2\alpha^2 - 3b}{2a^2}.$$

It follows that the Einstein–Maxwell equation on the pair $1.1^2.1$ has a solution in the case of signatures (4,0) and (2,2), and has no solutions in the case of signature (3,1), i.e., the pair $1.1^2.1$ is a Riemannian Einstein–Maxwell pair, as well as an Einstein–Maxwell pair of type (2,2).

References

[B] L. Berard Bergery, *Homogeneous Riemannian spaces of dimension four*, Seminar of A. Besse, Four-dimensional Riemannian geometry (1985), Moscow, Mir. (Russian)

[Be] A. L. Besse, *Einstein Manifolds*, Springer–Verlag, Berlin, Heildelberg, 1987.

[DK] B. Doubrov and B. Komrakov, *Low-Dimensional pseudo-Riemannian homogeneous spaces*, Preprint University of Oslo, **No. 13** (March 1995).

[F] D. Fuks, *Cohomology of infinite dimensional Lie algebras*, Moscow, Nauka, 1984. (Russian)

[I] S. Ishihara, *Homogeneous Riemannian spaces of four dimensions*, Jour. of the Math. Soc. of Japan, **7** (1955), 345–370.

[J] G. R. Jensen, *Homogeneous Einstein spaces of dimension four*, J. Diff. Geom., **3** (1969), 309–349.

[K] B. Komrakov, Jnr., *Four-dimensional pseudo-Riemannian homogeneous spaces. Classification of complex pairs*, Preprint University of Oslo, **No. 34** (December 1993).

[K1] B. Komrakov, Jnr., *Four-dimensional pseudo-Riemannian homogeneous spaces. Classification of complex pairs* II, Preprint University of Oslo, **No. 25** (May 1995).

[K2] B. Komrakov, Jnr., *Four-dimensional pseudo-Riemannian homogeneous spaces. Classification of real pairs*, Preprint University of Oslo, **No. 32** (June 1995).

[KT] B. Komrakov, A. Tchourioumov and others, *Three-dimensional isotropically-faithful homogeneous spaces*, Preprint University of Oslo, **No. 35–37** (Nov. 1993).

[KTD] B. Komrakov, A. Tchourioumov, B. Doubrov, *Two-dimensional homogeneous spaces*, Preprint University of Oslo, **No. 17** (June 1993).

[K–N] S. Kobayashi and K. Nomizu, *Foundations of differentiable geometry*, vol. No. 2, Interscience, Wiley, New York, 1969.

[L1] S. Lie, *Theorie der Transformationsgruppen.* III. Bestimmung aller Gruppen einer zweifach ausgedehnten Punktmannigfaltigkeit, Arch. for Math., Bd. III (1878), Kristiania, 93–165.

[L2] S. Lie, *Theorie der Transformationsgruppen. Ab.* III, Teubner, Leipzig, 1893.

[M] G. Mostow, *The extensibility of local Lie groups of transformations and groups on surfaces*, Ann. of Math., **32** (1950).

[W] J. Wolf, *The geometry and structure of isotropy irreducible homogeneous spaces*, Acta Math., **152** (1984), 141–142.

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