ON THE ANALYTICITY OF THE SCHWARZ OPERATOR WITH RESPECT TO A CURVE

LUCA PRECISO AND SERGEI V. ROGOSIN

ABSTRACT. We consider the Schwarz operator $\mathbf{T}[\cdot,\cdot]$ which assigns to each pair (ϕ, f), where ϕ is a plane curve and f is a real-valued function, a holomorphic function F in the domain D enclosed by ϕ such that boundary values of $\Re F$ coincide with f. We show that $\mathbf{T}[\phi, f]$ depends real analytically on (ϕ, f) in a suitable sense and in a Schauder space setting and we compute the first variations of $\mathbf{T}$ with respect its variables.

1. Introduction

The classical Schwarz boundary value problem (see [8]) consists of searching for a holomorphic function F in a (bounded or unbounded) plane domain D by given boundary values f of the real part $u = \Re F$ of the function F. It is known ([8, p. 210], [18, §30]) that the solution of such problem is determined uniquely up to a pure imaginary constant whenever f belongs to a suitable function space (e.g., f is Hölder-continuous), and the boundary ∂D is a smooth simple curve. Analogous result takes place for D being a multiply connected domain. The operator which assigns to each given function f the solution of the Schwarz problem (having prescribed imaginary part in a given point) is called the Schwarz operator (see [8, p. 208]).

The Schwarz operator $\mathbf{T}$ depends in fact not only on the boundary data f but also on the curve $\phi : \mathbb{T} \to \partial D$ encircling the domain D. Then the operator $\mathbf{T}$ is a map of two independent variables ϕ and f.

The aim of our paper is to carry out the perturbation analysis of the Schwarz problem, *i.e.* to study the regularity of the dependence of

Date: May 2, 2008.

1991 *Mathematics Subject Classification.* 30E20, 30E25, 45G05, 47H30.

Key words and phrases. Schwarz boundary value problem, nonlinear operators, simply connected domain, Schauder spaces.

The authors are indebted to the 'Gruppo Nazionale per l'Analisi Mat., la Probabilita' e loro Applicaz.' of the Italian 'Istituto Nazionale di Alta Matematica' "F.Severi" and to the 'Belarusian Fund for Fundamental Scientific Research' for their financial support.

$\mathbf{T}[\phi, f]$ on the functional variable ϕ and f, in a Schauder space setting, .

In the case of some simple curves the Schwarz operator can be represented in the explicit form. If, for instance, $\partial D = \mathbb{T}$, i.e., ∂D is the unit circle with the standard parametrization, then the Schwarz operator has the following representation:

circle (1.1)
$$(\mathbf{T}[\phi, f])(z) \equiv \frac{1}{2\pi i} \int\limits_{\mathbb{T}} f(\sigma) \frac{\sigma + z}{\sigma - z} \frac{d\sigma}{\sigma},$$

where $\phi \equiv id_{\mathbb{T}}$, i.e., ϕ is the identity map on $\mathbb{T}$ and $\Im\, \mathbf{T}[\phi, f](0) = 0$. For a general simply connected domain $D \equiv \mathbb{I}[\phi]$ enclosed by the Lyapunov curve ϕ (*i.e.* of class $C^{1,\alpha}$), the Schwarz operator is determined up to an imaginary constant by the formula

dom_s (1.2)
$$(\mathbf{T}[\phi, f])(z) \equiv \frac{1}{2\pi} \int\limits_{\phi(\mathbb{T})} f(\sigma) \frac{\partial}{\partial n_\sigma} \left[log \frac{\omega(z) - \omega(\sigma)}{1 - \overline{\omega(\sigma)} \omega(z)} \right] |d\sigma|,$$

(see *e.g.* [20, p.30]) where $\omega(\cdot)$ is a conformal mapping of the domain $\mathbb{I}[\phi]$ onto the unit disc $\mathbb{U}$ and the normal derivative eliminates the multivaluedness of the complex logarithm. Thus the explicit representation can be found for those domains which are mapped explicitly onto the unit disc.

The study of the regularity of certain nonlinear operators with respect to functional variables is quite intensive in the recent years. The questions we are discussing here are close to that on the regularity of certain families of curves [26]. The latter are motivated by the application in fluid mechanics (see, e.g., [25], [9]).

Among the results similar to ours we have to mention the classical theorem by Rado [24], which asserts the continuity of the dependence of the Riemann Map of a simply connected Jordan domain upon the boundary curve in the topology of the uniform convergence. More recently, Coifman and Meyer [6] have proved the analyticity of a nonlinear operator associated to the conformal representation of an unbounded simply connected domain having arc-length parametrized boundary with the direction of the tangent vector described by a function of class BMO. Later Wu [27] with advice of Coifman and with ideas of [6] has obtained two analyticity statements for bounded domains with arc-length parametrized boundary having certain symmetries.

By using a PDE approach and in the frame of Schauder spaces, Lanza [13] has shown the analyticity of the operator $\mathbf{h} : (\phi, w) \mapsto g^{(-1)}_{\phi,w} \circ \phi$, where $g_{\phi,w}$ is the conformal mapping of the unit disc $\mathbb{U}$ onto the bounded domain $\mathbb{I}[\phi]$ enclosed by ϕ, which is normalized by the conditions $g_{\phi,w}(0) = w \in \mathbb{I}[\phi]$, $g'_{\phi,w}(0) > 0$. Later it has appeared that the integral equation approach is more suitable to the study of the operator $\mathbf{h}$ even in the case of doubly connected domains.

Lanza and Rogosin have discovered a system of integral-functional equations for the operator $\mathbf{h}$ in the case of simply connected domains [16] and doubly connected domains [17] and have shown on the base of these systems that the operator $\mathbf{h}$ is real analytic in a Schauder space setting.

The systems of integral-functional equations mentioned above contain the Cauchy integral operator

Cauchy

$$(\mathbf{C}[\phi, f])(s) \equiv \frac{1}{2\pi i} p.v. \int_{\mathbb{T}} \frac{f(\sigma)\phi'(\sigma)}{\phi(\sigma) - \phi(s)} d\sigma. \tag{1.3}$$

In connection with the study of the regularity of operators of the type (1.3) we mention the contribution by Calderon, Coifman, Meyer, McIntosh, David. Calderon [3, Thm. 1] has shown that if ϕ is a graph of a Lipschitz function ψ, i.e., if $\phi(x) = x + i\psi(x)$ with $\psi' \in L^\infty(\mathbb{R})$ and if $\|\psi'\|_{L^\infty(\mathbb{R})} < \varepsilon$ for some $\varepsilon > 0$, then the linear integral operator with singular kernel $\frac{\phi'(y)}{\phi(y)-\phi(x)}$ is an element of the space $\mathcal{L}\left(L^2(\mathbb{R}, \mathbb{C}), L^2(\mathbb{R}, \mathbb{C})\right)$ of linear and continuous operators of $L^2(\mathbb{R}, \mathbb{C})$ to itself. Then, by using a standard argument of truncated kernel, one can deduce the analytic dependence of the integral operator with kernel $\frac{1}{\phi(y)-\phi(x)}$ upon ψ' with $\|\psi'\|_{L^\infty(\mathbb{R})} < \sigma$ with σ possibly less than 1 (cf., e.g., [5, p. 438]). Later, Coifman, McIntosh and Meyer [4, Thm. 1], and David by different method, [7, p. 178], have extended the validity of the same analyticity result to the case in which $\|\psi'\|_{L^\infty(\mathbb{R})} < 1$.

To study the regularity of the dependence of $\mathbf{T}[\phi, f]$ on ϕ and f we need to deal with functional variables having domains independent of ϕ. Then we study the regularity of some variants of the Schwarz operator. First we consider the modified Schwarz operator which maps a simple closed curve ϕ with nonvanishing derivative and a real valued function f of $\mathbb{T}$ to $\mathbf{T}[\phi, p \circ \phi^{(-1)}] \circ \phi$. We represent the modified Schwarz operator as a composition of operators whose regularity is known. Then we prove the real analiticity of the modified Schwarz operator by using the corresponding results for the singular (Cauchy) integral operator (as stated in [15, Thm 3.16]), and for the operator $\mathbf{h} : (\phi, w) \mapsto h[\phi, w] = g^{(-1)}_{\phi,w} \circ \phi$ (as stated in [16, Thm. 5.8]), where $g_{\phi,w} : \mathbb{U} \to \mathbb{I}[\phi]$ is the Riemann Map of the unit disc $\mathbb{U}$ onto the domain $\mathbb{I}[\phi]$, which is normalized by the conditions $g_{\phi,w}(0) = w, g'_{\phi,w}(0) > 0$. Moreover we calculate explicitly the first differential of the modified

Schwarz operator. The regularity of other variants of the Schwarz operator is considered but not all of them are real analytic.

The paper is organized as follows. In Section 2 we present basic notation and auxiliary results. In Section 3 we introduce the modified Schwarz operator. We show here the real analyticity of the modified Schwarz operator and calculate its first differential. Section 4 is devoted to the study of the regularity of another variant of the classical Schwarz operator.

2. Preliminaries and notation

For standard definitions of calculus on normed spaces we refer, e.g., to Prodi and Ambrosetti [23] and to Berger [2]. In particular, a finite product of normed spaces is endowed with the sup-norm of the norms of its components. Further, any complex normed space can be viewed as a real normed space. Accordingly, we will say that a map between two complex normed spaces is real differentiable, real analytic or real linear, respectively, if such a map has the corresponding property as a map between the underlined real normed spaces. In a contrary, if we are retaining the complex structure then we call a map with the corresponding property as complex differentiable, complex analytic or complex linear.

The inverse function of a function f is denoted $f^{(-1)}$ as opposed to the reciprocal of a real- or complex-valued function g, which is denoted by g^{-1}.

We denote by $\mathbb{U}$ the open unit disc in $\mathbb{C}$ (or in $\mathbb{R}^2$), by $\mathbb{T}$ the boundary of $\mathbb{U}$. The unit circle $\mathbb{T}$ is usually counter-clockwise oriented. For any set $D \subseteq \mathbb{R}^n$ we denote by $\mathrm{cl}\, D$ its closure, and by $\mathrm{int}\, D$ its interior.

The symbol $\Re$ ($\Im$) denotes the function which takes a complex number z to its real (imaginary) part. If $\mathbf{T}$ is an operator into a subset of complex-valued functions, then $\Re\mathbf{T}$ ($\Im\mathbf{T}$) is the operator which maps f to $\Re \circ \mathbf{T}[f]$ ($\Im \circ \mathbf{T}[f]$).

Let Ω be an open subset of $\mathbb{R}^n$. Let $\mathbb{N}$ be the set of nonnegative integers including 0, and let $m \in \mathbb{N}$. We denote by $\mathcal{C}^m(\Omega, \mathbb{C})$ the set of m-times continuously real differentiable complex-valued functions on Ω and by $\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathbb{C})$ the subspace of those functions $f \in \mathcal{C}^m(\Omega, \mathbb{C})$ such that, for all $\eta \in \mathbb{N}^n$ with $|\eta| = \eta_1 + \ldots + \eta_n \leq m$, the functions $D^\eta f \equiv \frac{\partial^{|\eta|} f}{\partial^{\eta_1} x_1 \ldots \partial^{\eta_n} x_n}$ can be continuously extended to $\mathrm{cl}\,\Omega$. If Ω is a bounded open subset of $\mathbb{R}^n$, we equip $\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathbb{C})$ with the norm $\|f\|_{\mathcal{C}^m(\mathrm{cl}\,\Omega,\mathbb{C})} \equiv \sum_{|\eta| \leq m} \sup_{\mathrm{cl}\,\Omega} |D^\eta f|$. The subspace of $\mathcal{C}^m(\mathrm{cl}\,\Omega, \mathbb{C})$ of those functions whose m-th order derivatives are α-Hölder continuous with exponent $\alpha \in]0,1]$ is denoted by $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{C})$. We endow $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{C})$ by the usual norm

$$\|f\|_{\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{C})} \equiv \sum_{j=0}^{m} \sup_{\mathrm{cl}\,\Omega} |D^j f| + \sup\left\{ \frac{|D^m f(s) - D^m f(t)|}{|s-t|^\alpha} : s,t \in \mathrm{cl}\,\Omega, s \neq t \right\}.$$

The spaces $\left(\mathcal{C}^m(\mathrm{cl}\,\Omega,\mathbb{C}); \|f\|_{\mathcal{C}^m(\mathrm{cl}\,\Omega,\mathbb{C})}\right)$ and $\left(\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{C}); \|f\|_{\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{C})}\right)$ are complex Banach spaces. If it clear on which domain these spaces are defined then we will use the following short notation $\|\cdot\|_m$ and $\|\cdot\|_{m,\alpha}$ for corresponding norms. If $n=2$ we denote by $\mathcal{H}(\Omega)$ the set of the holomorphic functions of Ω. We denote (see e.g., [15, p. 367]) $c[\Omega] \equiv \sup\{\frac{\lambda(x,y)}{|x-y|} : x,y \in \Omega, x \neq y\}$, where $\lambda(x,y) \equiv \inf\{\text{length of } \gamma \in \mathcal{C}^1([0,1],\Omega) : \gamma(0)=x, \gamma(1)=y\}$. The open subset Ω of $\mathbb{R}^n$ is said to be regular in sense of Whitney (or Whitney regular set), if Ω is bounded and connected, and if $c[\Omega] < +\infty$. It is well known (cf., *e.g.*, Jones [11, p. 73]) that if Ω is a bounded, connected, open subset of $\mathbb{R}^n$ of class $\mathcal{C}^1$, then it is Whitney regular set.

Whitney

Lemma 2.1. *Let $m,n,r,h \in \mathbb{N}$, let $n,r,h \geq 1$, and let $\alpha,\beta \in]0,1]$. Let Ω be an open Whitney's regular subset of $\mathbb{R}^n$. Then the following holds.*
(i) $\mathcal{C}^{m+1}(\mathrm{cl}\,\Omega,\mathbb{C})$ *is continuously imbedded in* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{C})$.
(ii) *The pointwise product in* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{C})$ *is continuous and* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{C})$ *is with this product a commutative Banach algebra with unity.*
(iii) *The reciprocal map in* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{R})$, *which maps a nonvanishing function f to its reciprocal f^{-1}, is complex analytic from the open subset* $\{f \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{C}) : f(x) \neq 0, \text{ for all } x \in \mathrm{cl}\,\Omega\}$ *of* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{R})$ *to itself.*
(iv) *Let Ω_1 be an open subset of $\mathbb{R}^r$, regular in sense of Whitney. If $F \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega_1,\mathbb{R}^h)$ and if $G \in \mathcal{C}^{m,\beta}(\mathrm{cl}\,\Omega,\mathrm{cl}\,\Omega_1)$, then $F \circ G \in \mathcal{C}^{m,\gamma_m(\alpha,\beta)}(\mathrm{cl}\,\Omega,\mathbb{R}^h)$, with $\gamma_0(\alpha,\beta)=\alpha\beta$, and $\gamma_m(\alpha,\beta) = min\ \{\alpha,\beta\}$ if $m>0$.*
(v) *Let $m \in \mathbb{N}, m \geq 1$. If $G \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{R}^n)$ is injective and satisfies the condition $DG \neq 0$ for all $x \in \mathrm{cl}\,\Omega$, then $G(\Omega)$ is a bounded connected open subset of $\mathbb{R}^n$, $G(\mathrm{cl}\,\Omega) = \mathrm{cl}\,G(\Omega)$, $c[G(\Omega)] < +\infty$, and $G^{(-1)} \in \mathcal{C}^{m,\alpha}(\mathrm{cl}\,G(\Omega),\mathrm{cl}\,\Omega)$.*

We refer for the proof of this Lemma to [15, Lem. 2.3].

We now define the Schauder spaces on plane Jordan curves, which are particular compact subsets of $\mathbb{C}$ with no isolated points. With somewhat more generality, we define the Schauder spaces on a general compact subset K of $\mathbb{C}$ with no isolated points. We say that a function f of K to $\mathbb{C}$ is complex differentiable at $z_0 \in \mathbb{C}$ if $\lim_{K \ni z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$

exists finite. We denote such limit by $f'(z_0)$. As usual the higher order derivatives, if they exist, are defined inductively. Let $m \in \mathbb{N}$. We denote by $C^m_*(K, \mathbb{C})$ the complex normed space of the m-times continuously complex differentiable functions f of K to $\mathbb{C}$ endowed with the norm $\|f\|_{C^m_*(K,\mathbb{C})} = \sum_{j=0}^m \sup_K |f^{(j)}|$. If $\alpha \in]0,1]$, we denote by $C^{m,\alpha}_*(K, \mathbb{C})$ the subspace of $C^m_*(K, \mathbb{C})$ of those functions having α-Hölder continuous m-th order derivative in K. If $f \in C^{0,\alpha}_*(K, \mathbb{C})$, then we set $|f : K|_\alpha \equiv \sup\left\{\frac{|f(z_1)-f(z_2)|}{|z_1-z_2|^\alpha} : z_1, z_2 \in K,\ z_1 \neq z_2\right\}$. We endow $C^{m,\alpha}_*(K, \mathbb{C})$ with the norm $\|f\|_{C^{m,\alpha}_*(K,\mathbb{C})} \equiv \|f\|_{C^m_*(K,\mathbb{C})} + |f^{(m)} : K|_\alpha$. If $B \subseteq \mathbb{C}$, we set $C^{m,\alpha}_*(K, B) \equiv \{f \in C^{m,\alpha}_*(K, \mathbb{C}) : f(K) \subseteq B\}$. If B is an open subset of $\mathbb{C}$ a straighforward compactness argument implies that $C^{m,\alpha}_*(K, B)$ is an open subset of $C^{m,\alpha}_*(K, \mathbb{C})$.

We denote by $C^{m,\alpha,0}_*(K, \mathbb{C})$ the closure of $C^\infty_*(K, \mathbb{C})$ in $C^{m,\alpha}_*(K, \mathbb{C})$. Then the following variant of [12, Cor. 4.24, Prop. 4.29] holds (cf. [15, Lem. 2.5].)

AK **Lemma 2.2.** *The following statements hold.*
(i) *Let* $\phi \in C^1_*(\mathbb{T}, \mathbb{C})$. *Then* $l_{\mathbb{T}}[\phi] \equiv \inf\left\{\frac{|\phi(x)-\phi(y)|}{|x-y|} : x, y \in \mathbb{T},\ x \neq y\right\} > 0$ *if and only if* ϕ *is injective and* $\phi'(\xi) \neq 0$ *for all* ξ *in* $\mathbb{T}$.
(ii) *The function of* $C^1_*(\mathbb{T}, \mathbb{C})$ *to* $\mathbb{R}$ *which maps* ϕ *to* $l_{\mathbb{T}}[\phi]$ *is continuous, and in particular, the set*

Def.Z (2.3) $$\mathcal{Z} \equiv \{\phi \in C^1_*(\mathbb{T}, \mathbb{C}) : l_{\mathbb{T}}[\phi] > 0\}$$

is open in $C^1_*(\mathbb{T}, \mathbb{C})$.

We have to note that a smooth curve is often defined as an equivalence class of smooth parametrizations. However, for our purposes, it is necessary to distinguish among the different parametrizations. Thus we define a (closed) curve of class $\mathcal{C}^{m,\alpha}$ on the complex plane $\mathbb{C}$ to be a map ϕ belonging to $C^{m,\alpha}_*(\mathbb{T}, \mathbb{C})$.

By a simple curve of class $\mathcal{C}^{m,\alpha}$ we understand a curve ϕ of class $\mathcal{C}^{m,\alpha}$ and belonging to $\mathcal{Z}$. A curve ϕ should not be confused with its locus $\phi(\mathbb{T})$. Our perturbation analysis of the Schwarz problem will deal with domains enclosed by a simple curve ϕ of class $\mathcal{C}^{m,\alpha}$ with $m \geq 1, \alpha \in]0,1[$ (*i.e.* a Lyapunov curve). Let $\phi \in \mathcal{Z}$. By $\mathbb{I}[\phi]$ we denote the bounded connected component of $\mathbb{C} \setminus \phi(\mathbb{T})$. Let $w \in \mathbb{I}[\phi]$. Let $g_{\phi,w}$ be the Riemann map of $\mathbb{U}$ to $\mathbb{I}[\phi]$ normalized by the conditions

Norm.RM (2.4) $$g_{\phi,w}(0) = w,\ g'_{\phi,w}(0) > 0.$$

The same symbol $g_{\phi,w}$ will be used also for the unique continuous extension to $\mathrm{cl}\,\mathbb{U}$. It is well known that $g_{\phi,w}$ is a homeomorphism between

$\mathrm{cl}\,\mathbb{U}$ and $\mathrm{cl}\,\mathbb{I}[\phi]$ (cf. *e.g.* [22, Thms. 3.5, 3.6]). A classical Warschawski result (cf. *e.g.* [22, Thm. 2.6]) implies that

Reg.RM (2.5)
$$g_{\phi,w} \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{U}, \mathbb{C}), \; g'_{\phi,w}(z) \neq 0 \; \forall z \in \mathrm{cl}\,\mathbb{U}$$

whenever $\phi \in C^{m,\alpha}_*(\mathbb{T}, \mathbb{C}) \cap \mathcal{Z}$ with $m \in \mathbb{N} \setminus \{0\}$ and $\alpha \in]0,1[$.

We are now ready to state the following, which collects a few facts which we need on the spaces $C^{m,\alpha}_*(K, \mathbb{C})$. For a proof and for appropriate references, we refer to [15, Lems. 2.7, 2.8].

schauder **Lemma 2.6.** *Let $m \in \mathbb{N}$, $\alpha, \beta \in]0,1]$, $\phi \in \mathcal{Z}$, $L = \phi(\mathbb{T})$. Then the following statements hold.*
(i) *$C^{m+1}_*(L, \mathbb{C})$ is continuously imbedded in $C^{m,\alpha}_*(L, \mathbb{C})$. If $\alpha < \beta$, then $C^{m,\beta}_*(L, \mathbb{C})$ is continuously imbedded in $C^{m,\alpha}_*(L, \mathbb{C})$.*
(ii) *The pointwise product is continuous in the Banach space $C^{m,\alpha}_*(L, \mathbb{C})$.*
(iii) *The reciprocal map $\mathbf{R}$ in $C^{m,\alpha}_*(L, \mathbb{C})$, which maps a nonvanishing function f to its reciprocal, is complex analytic from the open subset $C^{m,\alpha}_*(L, \mathbb{C} \setminus \{0\})$ of $C^{m,\alpha}_*(L, \mathbb{C})$ to itself. Moreover $D\mathbf{R}[f_0][\mu] = -\frac{\mu}{f_0^2}$ for all $f_0 \in C^{m,\alpha}_*(L, \mathbb{C} \setminus \{0\})$ and $\mu \in C^{m,\alpha}_*(L, \mathbb{C})$.*
(iv) *Let $\phi_1 \in \mathcal{Z}$, $L_1 = \phi_1(\mathbb{T})$. If $f \in C^{m,\alpha}_*(L_1, \mathbb{C})$ and if $g \in C^{m,\beta}_*(L, L_1)$, then $f \circ g \in C^{m,\gamma_m(\alpha,\beta)}_*(L, \mathbb{C})$ with $\gamma_0(\alpha, \beta) = \alpha\beta$ and $\gamma_m(\alpha, \beta) = \min\{\alpha, \beta\}$ if $m > 0$.*
(v) *Let $m \geq 1$. If $g \in C^{m,\alpha}_*(L, \mathbb{C})$ is injective and satisfies condition $g'(\xi) \neq 0$, for all $\xi \in L$, then $g^{(-1)} \in C^{m,\alpha}_*(g(L), L)$.*
(vi) *If $\mathbb{I}[\phi]$ denotes the open bounded and connected component of $\mathbb{C} \setminus \phi(\mathbb{T})$, then $\partial\mathbb{I}[\phi] = \phi(\mathbb{T})$.*
(vii) *Let $\phi \in \mathcal{C}^{m,\alpha}_*(\mathbb{T}, \mathbb{C}) \cap \mathcal{Z}$. Then the trace operator $\mathbf{R}$ from $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{I}[\phi], \mathbb{C})$ to $\mathcal{C}^{m,\alpha}_*(\phi(\mathbb{T}), \mathbb{C})$ defined by $\mathbf{R}[F] = F_{/\phi(\mathbb{T})}$ is complex linear and continuous.*
(viii) *If $f \in \mathcal{Z}$, and if $f(\mathbb{T}) \subseteq \mathbb{T}$, then $f(\mathbb{T}) = \mathbb{T}$ and f is a homeomorphism of $\mathbb{T}$ to itself.*

The following Theorem collects known facts about the singular (Cauchy) integral and the Schwarz operator for the unit circle (cf. *e.g.* [8, §3-5, pp. 44-46, 208-210, 248]).

Schw **Theorem 2.7.** *Let $m \in \mathbb{N}$, $\alpha \in]0,1[$. Then the following statements hold.*
(i) *Let $\phi \in C^{1,\alpha}_*(\mathbb{T}, \mathbb{C}) \cap \mathcal{Z}$, $L = \phi(\mathbb{T})$. Then for all $f \in C^{m,\alpha}_*(L, \mathbb{C})$, the singular integral*

sch (2.8)
$$\mathbf{S}_\phi[f](\tau) \equiv \frac{\mathrm{p.v.}}{\pi i} \int_\phi \frac{f(\sigma)}{\sigma - \tau}\, d\sigma = \frac{\mathrm{p.v.}}{\pi i} \int_{\mathbb{T}} \frac{f(\phi(\eta))\phi'(\eta)}{\phi(\eta) - \tau} d\eta,$$

exists in the sense of the principal value for all $\tau \in L$*, and* $\mathbf{S}_\phi[f](\cdot) \in C_*^{m,\alpha}(L,\mathbb{C})$*. The operator* $\mathbf{S}_\phi$ *defined by* (2.8) *is linear and continuous from* $C_*^{m,\alpha}(L,\mathbb{C})$ *to itself and the equality* $(\mathbf{S}_\phi[g])' = \mathbf{S}_\phi[g']$ *holds for all* $g \in C_*^{1,\alpha}(L,\mathbb{C})$*. If* ϕ *coincides with the identity map* $\mathrm{id}_{\mathbb{T}}$*, then we set* $\mathbf{S} \equiv \mathbf{S}_\phi$*.*

(ii) *Let* $\mathrm{ind}\,[\phi]$ *be the index of the curve* $\theta \mapsto \phi(e^{i\theta})$*,* $\theta \in [0,2\pi]$ *with respect to any of the points of* $\mathbb{I}[\phi]$*. Then for all* $f \in C_*^{m,\alpha}(L,\mathbb{C})$*, the Cauchy type integral* $\mathbf{C}_\phi[f]$ *of* $\mathbb{C}\setminus\{L\}$ *to* $\mathbb{C}$ *defined by*

$$\mathbf{C}_\phi[f](z) \equiv \frac{\mathrm{ind}\,[\phi]}{2\pi i}\int_\phi \frac{f(\sigma)}{\sigma - z}\,d\sigma, \qquad \forall z \in \mathbb{C}\setminus L,$$

is holomorphic. The function $\mathbf{C}_\phi[f]_{|\mathbb{I}[\phi]}$ *admits a continuous extension to* $\mathrm{cl}\,\mathbb{I}[\phi]$*, which we still denote by* $\mathbf{C}_\phi[f]$ *and we have* $\mathbf{C}_\phi[f] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{I}[\phi],\mathbb{C}) \cap H(\mathbb{I}[\phi])$*. Moreover the Plemelj formula* $\mathbf{C}_\phi[f](\tau) = \frac{1}{2}f(\tau) + \frac{\mathrm{ind}\,[\phi]}{2}\mathbf{S}_\phi[f](\tau)$ *for all* $\tau \in L$ *hold and* $\mathbf{C}_\phi^+[\cdot]$ *defines a linear and continuous operator of* $C_*^{m,\alpha}(L,\mathbb{C})$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I}[\phi],\mathbb{C})$*. If* ϕ *coincides with the identity map* $\mathrm{id}_{\mathbb{T}}$*, then we set* $\mathbf{C} \equiv \mathbf{C}_\phi$*.*

(iii) *For all* $f \in \mathcal{C}_*^{m,\alpha}(\mathbb{T},\mathbb{C})$*, the function defined by*

Schw1 (2.9)
$$\mathbf{T}[f](z) \equiv \frac{1}{2\pi i}\int\limits_{\mathbb{T}} f(t)\frac{t+z}{t-z}\frac{dt}{t}, \ \ \forall z \in \mathbb{U},$$

and extended by continuity to $\mathrm{cl}\,\mathbb{U}$ *belongs to* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{U},\mathbb{C}) \cap \mathcal{H}(\mathbb{U})$*. This operator is called the Schwarz operator for the unit disc, while the integral in formula (2.9) is called the Schwarz integral for the unit disc. Moreover the following formulas connect the Schwarz operator to the Cauchy type operator and the singular (Cauchy) integral*

Schw3 (2.10)
$$\mathbf{T}[f](z) = 2\mathbf{C}[f](z) - \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f(t)dt}{t}, \ \ \forall z \in \mathbb{U},$$

Schw5 (2.11)
$$\mathbf{T}[f](\tau) = f(\tau) + \mathbf{S}[f](\tau) - \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f(t)dt}{t} = f(\tau) - i\mathbf{H}[f](\tau), \ \ \tau \in \mathbb{T},$$

for all $f \in \mathcal{C}_*^{m,\alpha}(\mathbb{T},\mathbb{C})$ *where* $\mathbf{H}[f](\cdot)$ *is the singular integral operator with the Hilbert kernel*

Hilb (2.12)
$$\mathbf{H}[f](e^{is}) \equiv \frac{1}{2\pi}\int\limits_0^{2\pi} f(e^{i\sigma})\cot\frac{\sigma - s}{2}d\sigma.$$

In particular the operator $\mathbf{T}[\cdot]$ *is linear and continuos from* $\mathcal{C}^{m,\alpha}_*(\mathbb{T},\mathbb{R})$ *to* $\mathcal{C}^{m,\alpha}(\mathrm{cl}\,\mathbb{U},\mathbb{C})$.

(iv) *For all* $f \in \mathcal{C}^{m,\alpha}_*(\mathbb{T},\mathbb{R})$, *the function* $\mathbf{T}[f]$ *determined by the formula (2.9) is a solution of the following boundary value problem*

Schw2

$$\Re\, F(\tau) = f(\tau),\ \tau \in \mathbb{T}, \tag{2.13}$$

and satisfies $\Im \mathbf{T}[f](0) = 0$. *The general solution of (2.13) in* $\mathcal{C}^0(\mathrm{cl}\,\mathbb{U},\mathbb{C}) \cap \mathcal{H}(\Omega)$ *is* $F = \mathbf{T}[f] + ic$, *where* $c \in \mathbb{R}$.

3. Real analyticity of some variants of the Schwarz operator

In this section we carry out the perturbation analysis of the Schwarz problem in a simply connected domain in the case of domains enclosed by Lyapunov curves and of H older continuous boundary data. In this Schauder space setting we obtain the real analyticity of some variants of the Schwarz operator and we compute the first differential of one of these operators.

Let ϕ be a simple (closed) curve of class $C^{1,\alpha}$ with $\alpha \in]0,1[$, $f \in C^{0,\alpha}_*(\phi(\mathbb{T}),\mathbb{R})$ and let $w \in \mathbb{I}[\phi]$. *The classical Schwarz problem* consists in searching for a complex-valued function of $\mathrm{cl}\,\mathbb{I}[\phi]$, holomorphic in $\mathbb{I}[\phi]$ and such that

Sch.bvp

$$\begin{cases} F \in C^0(\mathrm{cl}\,\mathbb{I}[\phi],\mathbb{C}) \cap \mathcal{H}(\mathbb{I}[\phi]), \\ \Re\, F(t) = f(t),\ \forall t \in \phi(\mathbb{I}),\quad \Im\, F(w) = 0. \end{cases} \tag{3.1}$$

Let $g_{\phi,w}$ be the Riemann map of $\mathbb{I}[\phi]$ as in (2.4). By composition with $g_{\phi,w}$ it is not difficult to check that a function F satisfies problem (3.1) if and only if the function $\tilde{F} \equiv F \circ g_{\phi,w}$ satisfies the following problem in $\mathrm{cl}\,\mathbb{U}$

Sch.bvpU

$$\begin{cases} \tilde{F} \in C^0(\mathrm{cl}\,\mathbb{U},\mathbb{C}) \cap \mathcal{H}(\mathbb{U}), \\ \Re \tilde{F}_{|\mathbb{T}} = f \circ (g_{\phi,w_{|\mathbb{T}}}),\quad \Im \tilde{F}(0) = 0. \end{cases} \tag{3.2}$$

By (2.5), Lemma 2.6 (iv) and by Theorem 2.7 (iv), problem (3.2) has a unique solution $\tilde{F}$ and $\tilde{F} \equiv F \circ g_{\phi,w}$ is the unique solution of (3.1). By (2.10) F satisfies the following formula

Sol.Schw.

$$F(z) = \frac{1}{\pi i}\int_{\mathbb{T}} \frac{(f \circ g_{\phi,w})(\sigma)}{\sigma - g^{(-1)}_{\phi,w}(z)} d\sigma - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(f \circ g_{\phi,w})(\sigma)}{\sigma} d\sigma \tag{3.3}$$

Let $\mathbf{T}$ be the operator from $\{(\phi,f,w) \in (C^{1,\alpha}_*(\mathbb{T},\mathbb{C}\cap\mathcal{Z}) \times C^{0,\alpha}_*(\phi(\mathbb{T}),\mathbb{R}) \times \mathbb{C} : w \in \mathbb{I}[\phi]\}$ to $C^0(\mathrm{cl}\,\mathbb{I}[\phi],\mathbb{C})$ defined by

Def.Schw.op. (3.4)
$$\mathbf{T}[\phi, f, w] \equiv F,$$

where F is the unique solution of (3.1). We call this operator the (classical) Schwarz operator. We should note that the Schwarz problem is well-posed for classes of curves and boundary data more general that those we consider here but we are not going to deal with those classes. By (2.5), Lemma 2.6 (iv) and (vii), Theorem 2.7 (ii) and by Lemma 2.1 the following holds

Reg.Schw.sol. (3.5)
$$\mathbf{T}[\phi, f, w] \in C^{m,\alpha}(\mathrm{cl}\, \mathbb{I}[\phi], \mathbb{C}) \cap \mathcal{H}(\mathbb{I}[\phi]),$$

for all $\phi \in C^{m,\alpha}_{*}(\mathbb{T}, \mathbb{C}) \cap \mathcal{Z}$, $f \in C^{m,\alpha}_{*}(\phi(\mathbb{T}), \mathbb{R})$, $w \in \mathbb{I}[\phi]$ with $m \in \mathbb{N} \setminus \{0\}$ and $\alpha \in]0, 1[$.

To get regularity results about the dependence of $\mathbf{T}[\phi, f, w]$ on (ϕ, f, w), together with a description of the first variation with respect these variables, we have to tackle two difficulties. The first is about the choice of a smooth variant of the Schwarz operator. Indeed we cannot study directly the operator $\mathbf{T}$ because the functions f and $\mathbf{T}[\phi, f, w]$ are defined in domains depending on ϕ. Then we introduce the modified Schwarz operator which has functionals variables defined in fixed domains. The modified Schwarz operator is obtained from the Schwarz operator by describing the boundary data f and the solution $\mathbf{T}[\phi, f, w]$ with their composition with the curve ϕ. The second difficulty is to choose a suitable representation of the modified Schwarz operator involving operators having known regularity and whose first differentials can be explicitely calculated. To tackle this difficulty we prefer to use the representation (3.3) instead of (1.2). Indeed by using (3.3) the modified Schwarz operator can be written in terms of the singular (Cauchy) integral operator $\mathbf{C}$ and of the operator $\mathbf{h}$, and the regularity of both of them is known in Schauder spaces (see Introduction).

In order to prove an analyticity result for the modified Schwarz operator and to compute its first differential we need the following Lemma.

Diff. **Lemma 3.6.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1]$. Then $C^{m,\alpha}_{*}(\mathbb{T}, \mathbb{C} \setminus \{0\}) \cap \mathcal{Z}$ is an open subset of $C^{m,\alpha}_{*}(\mathbb{T}, \mathbb{C})$ and the operator $\mathbf{G}$ from $C^{m,\alpha}_{*}(\mathbb{T}, \mathbb{C} \setminus \{0\}) \cap \mathcal{Z}$ to $\mathbb{C}$ defined by $\mathbf{G}[h] \equiv \int_{h(\mathbb{T})} \frac{p \circ h^{(-1)}(\sigma)}{\sigma} d\sigma = \mathrm{Ind}[h] \int_{\mathbb{T}} \frac{p(\eta) h'(\eta)}{h(\eta)} d\eta$ is complex analytic. Moreover its first differential at $h_0 \in C^{1,\alpha}_{*}(\mathbb{T}, \mathbb{C} \setminus \{0\}) \cap \mathcal{Z}$ satisfies*

Diff.1 (3.7)

$$(D\mathbf{G}[h_0])[\mu] = -\int_{h_0(\mathbb{T})} \frac{\mu \circ h_0^{(-1)}(\sigma)}{\sigma}(p \circ h_0^{(-1)})'(\sigma))d\sigma = -\mathrm{Ind}[h_0]\int_{\mathbb{T}} \frac{p'\mu}{h_0} d\sigma$$

for all $\mu \in C_*^{m,\alpha}(\mathbb{T}, \mathbb{C})$.

Proof. Since $\mathbb{C} \setminus \{0\}$ is an open subset of $\mathbb{C}$ $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C} \setminus \{0\})$ is an open subset of $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C})$. Then by Lemma 2.2 (ii) and Lemma 2.6 (i), $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C}) \cap \mathcal{Z}$ and $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C} \setminus \{0\}) \cap \mathcal{Z}$ are an open subsets of $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C})$. Then the first By Lemma 2.6 (ii) and (iii) the operator $h \mapsto \frac{p(\cdot)h'(\cdot)}{h(\cdot)}$ is complex analytic from $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C} \setminus \{0\})$ to $C_*^{m-1,\alpha}(\mathbb{T}, \mathbb{C})$ and then $\mathbf{G}$ is complex analytic. By chain rule and Lemma 2.6 (iii) the formula $(D\mathbf{G}[h_0])[\mu] = -\int_{\mathbb{T}} \left[\frac{p\mu'}{h_0} + p\mu \frac{d}{d\eta}\left(\frac{1}{h_0}\right)\right] d\eta$ holds for all $\mu \in C_*^{m,\alpha}(\mathbb{T}, \mathbb{C})$. Formula (3.7) follows by integrating by parts the second addend of the integrand function just obtained and by carrying out the substitution $\eta = h_0^{(-1)}(\sigma)$. □

Then we have the following.

An.T* **Proposition 3.8.** *Let* $m \in \mathbb{N} \setminus \{0\}, \alpha \in]0, 1[$. *Let*

(3.9)

$$\mathcal{A}_*^{m,\alpha} \equiv \{(\phi, p, w) \in (C_*^{m,\alpha}(\mathbb{T}, \mathbb{C}) \cap \mathcal{Z}) \times C_*^{m,\alpha}(\mathbb{T}, \mathbb{R}) \times \mathbb{C} \,:\, w \in \mathbb{I}[\phi]\}.$$

Let $\mathbf{T}_*$ *be the operator of* $\mathcal{A}_*^{m,\alpha}$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{U}, \mathbb{C})$ *defined by* $\mathbf{T}_*[\phi, p, w] \equiv \mathbf{T}[\phi, p \circ \phi^{(-1)}, w]] \circ \phi$ *for all* $(\phi, p, w) \in \mathcal{A}_*^{m,\alpha}$. *Then the following statements hold.*
(i) *The set* $\mathcal{A}_*^{m,\alpha}$ *is an open subset of* $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C}) \times C_*^{m,\alpha}(\mathbb{T}, \mathbb{R}) \times \mathbb{C}$ *and the operator* $\mathbf{T}_*$ *is real analytic in its domain* $\mathcal{A}_*^{m,\alpha}$.
(ii) *Let* $(\phi_0, p_0, w_0) \in \mathcal{A}_*^{m,\alpha}$, $h_0 \equiv g_{\phi_0,w_0}^{(-1)} \circ \phi_0$, $k_0 \equiv h_0^{(-1)}$ *and let* $g_0 \equiv g_{\phi_0,w_0}$.

Then $\frac{\partial \mathbf{T}_*}{\partial p}[\phi_0, p_0, w_0][g] = \mathbf{T}_*[\phi_0, g, w_0]$ *for all* $g \in C_*^{m,\alpha}(\mathbb{T}, \mathbb{R})$ *and the following formulas for the partial derivatives with respect* ϕ *and* w *hold.*

DTphi (3.10)
$$\frac{\partial \mathbf{T}_*}{\partial \phi}[\phi_0,p_0,w_0][\mu](t)=$$
$$\frac{1}{\pi}\int_{\mathbb{T}}\Bigg[\frac{h_0(t)\Im(\mathbf{I}-\mathbf{S})\left[\frac{\Re(\mathbf{I}-\mathbf{S})[\mu\circ k_0]}{\mathrm{id}_{\mathbb{T}}g_0'}\right](h_0(t))-\sigma\Im(\mathbf{I}-\mathbf{S})\left[\frac{\Re(\mathbf{I}-\mathbf{S})[\mu\circ k_0]}{\mathrm{id}_{\mathbb{T}}g_0'}\right](\sigma)}{\sigma-h_0(t)}$$
$$(p\circ k_0)'(\sigma)\Bigg]d\sigma+$$
$$+\frac{1}{g_0'(0)\pi^2}\int_{\mathbb{T}}\Bigg[\frac{-h_0(t)\Re\left(\overline{h_0(t)}\int_{\mathbb{T}}\frac{\mu\circ k_0(\eta)}{\eta}d\eta\right)+\sigma\Re\left(\overline{\sigma}\int_{\mathbb{T}}\frac{\mu\circ k_0(\eta)}{\eta}d\eta\right)}{\sigma-h_0(t)}$$
$$(p\circ k_0)'(\sigma)\Bigg]d\sigma+$$
$$+\frac{1}{2\pi}\int_{\mathbb{T}}\Im(\mathbf{I}-\mathbf{S})\left[\frac{\Re(\mathbf{I}-\mathbf{S})[\mu\circ k_0]}{\mathrm{id}_{\mathbb{T}}g_0'}\right](\sigma)(p\circ k_0)'(\sigma)d\sigma+$$
$$-\frac{1}{2g_0'(0)\pi^2}\int_{\mathbb{T}}\Re\left(\overline{\sigma}\int_{\mathbb{T}}\frac{\mu\circ k_0(\eta)}{\eta}d\eta\right)(p\circ k_0)'(\sigma)d\sigma,$$

DTw (3.11)
$$\frac{\partial \mathbf{T}_*}{\partial w}[\phi_0,p_0,w_0](t)=\frac{2}{g_0'(0)\pi}\int_{\mathbb{T}}\frac{h_0(t)\Im(h_0(t))-\sigma\Im(\sigma)}{\sigma-h_0(t)}(p\circ k_0)'(\sigma)d\sigma+$$
$$+\frac{1}{g_0'(0)\pi}\int_{\mathbb{T}}\Im(\sigma)(p\circ k_0)'(\sigma)d\sigma,$$

for all $\mu\in C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$ *and for all* $t\in\mathbb{T}$.

Proof. Let $\mathcal{E}^{m,\alpha}\equiv\{(\phi,w)\in(C_*^{m,\alpha}(\mathbb{T},\mathbb{C})\cap\mathcal{Z})\times\mathbb{C}: w\in\mathbb{I}[\phi]\}$ and let $\mathbf{h}$ be the operator of $\mathcal{E}^{m,\alpha}$ to $C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$ defined by $\mathbf{h}[\phi,w]\equiv g_{\phi,w}^{(-1)}\circ\phi$ for all $(\phi,w)\in\mathcal{E}^{m,\alpha}$. By formula (3.3) and by Theorem (2.7) (ii) the boundary values of $\mathbf{T}_*[\phi_0,p_0,w_0]$ satisfies the following formula

An.T*1 (3.12)
$$\mathbf{T}_*[\phi,p,w](t)\equiv\mathbf{T}[\phi,p\circ\phi^{(-1)},w]]\circ\phi=$$
$$p(t)+\frac{\mathrm{p.v.}}{\pi i}\int_{\mathbb{T}}\frac{p\circ\mathbf{h}[\phi,w]^{(-1)}(\sigma)}{\sigma-\mathbf{h}[\phi,w](t)}d\sigma-\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{p\circ\mathbf{h}[\phi,w]^{(-1)}(\sigma)}{\sigma}d\sigma$$

for all $t\in\mathbb{T}$ and for all $(\phi,p,w)\in\mathcal{A}_*^{m,\alpha}$. In the notation of Lemma 2.7 (i) the second summand in (3.12) is equal to $\mathbf{S}_{\mathbf{h}[\phi,w]}[p\circ\mathbf{h}[\phi,w]^{(-1)}]$. By [15, Prop. 4.1] the operator $(\gamma,p)\mapsto\mathbf{S}_\gamma[p\circ\gamma^{(-1)}]\circ\gamma$ is (complex) analytic from $(C_*^{m,\alpha}(\mathbb{T},\mathbb{C})\cap\mathcal{Z})\times C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$ to $C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$. Moreover by

[16, Thm. 5.4] the operator $\mathbf{h}$ is real analytic from $\mathcal{E}_*^{m,\alpha}$ to $C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$. Since the sum of analytic operators is an analytic operator, the second summand in (3.12) depends real analytically on (ϕ,p,w). By Lemma 3.6 the third summand in (3.12) depends (complex) analytically on (ϕ,p). Then $\mathbf{T}_*$ is real analytic in its domain. We now prove statement (ii).

By [15, Prop. 4.1 (i)] and by Lemma 3.6 the following formula holds

An.T*2
$$\frac{\partial \mathbf{T}_*}{\partial \phi}[\phi_0,p_0,w_0][\mu](t) = \frac{1}{\pi i}\int_{\mathbb{T}} \frac{\frac{\partial \mathbf{h}}{\partial \phi}[\phi_0,w_0][\mu](t) - \frac{\partial \mathbf{h}}{\partial \phi}[\phi_0,w_0][\mu](h_0^{(-1)}(\sigma))}{\sigma - h_0(t)}(p\circ h_0^{(-1)})' d\sigma + \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\frac{\partial \mathbf{h}}{\partial \phi}[\phi_0,w_0][\mu](h_0^{(-1)}(\sigma))}{\sigma}(p\circ h_0^{(-1)})' d\sigma \tag{3.13}$$

for all $t\in\mathbb{T}$ and for all $\mu\in C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$. We now compute $\frac{\partial \mathbf{h}}{\partial \phi}[\phi_0,w_0][\mu]$ by using [16]. In [16] Lanza & Rogosin introduced a functional equation whose solution set coincides with the graph of $\mathbf{h}$. Precisely let

An.T*3
$$\mathcal{D} \equiv \Big\{ (\phi,w,h)\in C_*^{m,\alpha}(\mathbb{T},\mathbb{C})\times\mathbb{C}\times C_*^{m,\alpha}(\mathbb{T},\mathbb{C}) : \phi\in\mathcal{Z},\ w\in \mathbf{I}[\phi], \mathrm{Re}\left\{\frac{\mathrm{ind}[h]}{2\pi i}\int_{\mathbb{T}}\frac{\phi(s)h'(s)}{h^2(s)}\,ds\right\}>0\Big\}, \tag{3.14}$$

They proved that $\mathcal{D}$ is an open subset of $C_*^{m,\alpha}(\mathbb{T},\mathbb{C})\times\mathbb{C}\times C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$, and that the triple $(\phi,w,h)\in\mathcal{D}$ satisfies the operator equation

An.T*4
$$\mathbf{P}[\phi,w,h]=0, \tag{3.15}$$

with

An.T*5
$$\mathbf{P}[\phi,w,h]\equiv(\mathbf{P}_l[\phi,w,h])_{l=1,2,3,4}\equiv \Bigg(\mathrm{Re}\left\{\phi(\cdot)-\frac{\mathrm{ind}[h]}{\pi i}\,\mathrm{p.v.}\int_{\mathbb{T}}\frac{\phi(s)h'(s)}{h(s)-h(\cdot)}\,ds\right\}, \frac{\mathrm{ind}[h]}{2\pi i}\int_{\mathbb{T}}\frac{\phi(s)h'(s)}{h(s)}\,ds - w, \mathrm{Im}\left\{\frac{\mathrm{ind}[h]}{2\pi i}\int_{\mathbb{T}}\frac{\phi(s)h'(s)}{h^2(s)}\,ds\right\}, h(\cdot)\overline{h(\cdot)}-1\Bigg), \tag{3.16}$$

if and only if $h=\mathbf{h}[\phi,w]$. They proved that $\mathbf{P}$ is a real analytic operator from $\mathcal{D}$ to $C_*^{m,\alpha}(\mathbb{T},\mathbb{R})\times\mathbb{C}\times\mathbb{R}\times C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$ and they studed the solvability of the functional equation $\frac{\partial \mathbf{h}}{\partial \phi}[\phi_0,w_0][g]=B$ with $B\in C_*^{m,\alpha}(\mathbb{T},\mathbb{R})\times\mathbb{C}\times\mathbb{R}\times C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$. By differentiating $\mathbf{P}[\phi,w_0,\mathbf{h}[\phi,w_0]=$

0 with respect ϕ and [16, Lems. 3.11 and 4.1], for any $\mu \in C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$ $g = \frac{\partial \mathbf{h}}{\partial \phi}[\phi_0, w_0][\mu]$ is the unique solution of the functional equation

An.T*6

$$\frac{\partial \mathbf{P}}{\partial h}[\phi_0, w_0, h_0][g] = -\frac{\partial \mathbf{P}}{\partial \phi}[\phi_0, w_0, h_0][\mu] = \left(-\Re\left(\mu - \mathbf{S}[\mu \circ k_0] \circ h_0\right), \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\mu \circ k_0(\eta)}{\eta} d\eta, -\Im\left(\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\mu \circ k_0(\eta)}{\eta^2} d\eta\right).0\right) \tag{3.17}$$

By using the explicit solution of $\frac{\partial \mathbf{h}}{\partial \phi}[\phi_0, w_0][g] = B$ given in [16, Prop. 5.3] and by (3.18) the following equality follows

An.T*7

$$\frac{\partial \mathbf{h}}{\partial \phi}[\phi_0, w_0][\mu](k_0(s)) = is\left(\Im(\mathbf{I}-\mathbf{S})\left[\frac{\Re(\mathbf{I}-\mathbf{S})[\mu \circ k_0]}{\mathrm{id}_{\mathbb{T}} g_0'}\right](s) + \Im(\mathbf{I}-\mathbf{S})\left[\frac{\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\mu \circ k_0(\eta)}{\eta} d\eta,}{\mathrm{id}_{\mathbb{T}} g_0'}\right](s) + c\right), \tag{3.18}$$

for all $s \in \mathbb{T}$, where c is a complex constant independent of s.

Since

An.T*8

$$(\mathbf{I}-\mathbf{S})\left[\frac{1}{\mathrm{id}_{\mathbb{T}} g_{\phi_0,w_0}'}\right](s) = \frac{2}{g_0'(0)s}, \tag{3.19}$$

for all $s \in \mathbb{T}$ and since $\int_{\mathbb{T}} (p \circ h_0^{(-1)})' d\sigma = 0$ formula of (3.10) follows. By an analogous argument based again in the [16, Prop. 5.3] also formula (3.11)can be obtained. □

References

Akh [1] N.I.Akhiezer *Elements of the Theory of Elliptic Functions.* Amer. Math. Soc., Providence, Rhode Island (1990).

Berg [2] M.S.Berger *Nonlineariry and Functional Analysis.* Amer. Math. Soc., Providence, Rhode Island (1977).

Cald [3] A.P.Calderon (1977) Cauchy integrals on Lipschitz curves and related operators. *Proc. Nat. Acad. Sci. U.S.A.* **74**, 1324-1327.

CoMcIMe [4] R.R.Coifman, A.McIntosh & Y.Meyer (1982) L'integrale de Cauchy definit un operator borne sur L^2 pour la courbes lipschitziennes. *Ann. Math.* **116** (1972), 361-387.

CoiMe_non [5] R.R.Coifman & Y.Meyer (1983) L'anlyse armonique non lineaire. In *Proc. of a seminar held in Turin and Milan in 1982: Topics on modern harmonic analysis.* (De Lionida-Ricci, ed.), INDAM F.Severi, Roma **2**, 707-721.

CoiMe_lav [6] R.R.Coifman & Y.Meyer (1983) Lavrentiev's curves and conformal mappings. *Mittag-Leffler Institute, Stockholm*, Report No. 5.

Dav [7] G.David (1984) Operateurs integraux singuliers sur certaines courbes du plan complexe. *Ann. Sci. Ecole Norm Sup., 4^e serie* **17**, 157-189.

Gakh [8] F.D.Gakhov *Boundary Value Problems.* Pergamon Press, Oxford (1966).

Gust [9] B.Gustafsson & M.Sakai Some geometric properties of a Hele-shaw flow moving boundary problem. In *Free boundary value problems in fluid flow with applications* (Montreal, PQ 1990), 53-57. Pitman Research Notes Math. Ser. **282**. Longman Sci. Tech., Harlow (1993).

HillePh [10] E.Hille & R.S.Philips *Functional Analysis and Semigroups.* Amer. Math. Soc. Colloq. Publ. **31** (1957).

Jones [11] P.Jones (1981) Quasiconformal mappings and extendability of functions in Sobolev spaces. *Acta Math.* **147**, 71-88.

La91 [12] M.Lanza de Cristoforis (1991) Properties and pathalogies of the composition and inversion operators in Schauder spaces. *Acc. Naz. delle Sci. detta dei XL* **15**, fasc. 5, 93-109.

La97 [13] M.Lanza de Cristoforis (1997) A functional decomposition theorem for the conformal representation. *J. Math. Soc. Japan* **49**, 759-780.

La96 [14] M.Lanza de Cristoforis (2000) Differentiability properties of an abstract autonomous composition operator. *J. London Math. Soc.(2)* **61**, 923-936.

LaPre99 [15] M.Lanza de Cristoforis & L.Preciso (1999) On the analyticity of the Cauchy integral in Schauder spaces. *J. Int. Equat. Appl.* **11**, No 3, 363-391.

LaRo00 [16] M.Lanza de Cristoforis & S.V.Rogosin (2000) Analyticity of a nonlinear operator associated to the conformal representation in Schauder spaces. An integral equation approach. *Math. Nachr.* **220**, 59-77.

LaRo01 [17] M.Lanza de Cristoforis & S.V.Rogosin (2001) Analyticity of a nonlinear operator associated to the conformal representation of a doubly connected domain in Schauder spaces. *Complex Variables* **44**, 193-223.

Mikh [18] S.G.Mikhlin *Integral Equations.* Pergamon Press, New York (1964).

Mit98 [19] V.V.Mityushev (1998) Hilbert boundary value problem for multiply connected domains. *Complex Variables* **35**, 283-295.

MiR [20] V.V.Mityushev & S.V.Rogosin *Constructive Methods for Linear and Nonlinear Boundary Value Problems for Analytic Functions.* Chapman & Hall / CRC, Boca Raton - London etc. (2000).

Pomm [21] Ch.Pommerenke *Univalent Functions, with a chapter on quadratic differentials by Gerd Jensen.* Vandenhoeck & Ruprecht, Göttingen (1975).

Pomm92 [22] Ch.Pommerenke *Boundary behaviour of Conformal Map.* Springer Verlag, Berlin etc. (1992).

ProdiA [23] G.Prodi & A.Ambrosetti *Analisi non lineare.* Editrice Tecnico Scientifica, Pisa (1973).

Rado [24] T.Rado (1923) Sur la representation conforme de domains variables. *Acta Univ. Szeged* **1**, 180-186.

Sakai1 [25] M.Sakai *Quadrature domains.* Lecture Notes in Math. **934**. Springer Verlag, Berlin (1982).

Sakai [26] M.Sakai (1993) Regularity of boundaries of quadrature domains in two dimensions. *SIAM J. Math. Anal.* **24**, No 2, 341-364.

Wu [27] S.Wu (1993) Analytic dependence of Riemann mappings for bounded domains and minimal surfaces. *Comm. Pure Appl. Math.* **46**, 1303-1326.

LaPre01 [28] M.Lanza de Cristoforis & L.Preciso (2001) Differentiability properties of some nonlinear operators associated to the conformal welding of Jordan curves in Schauder spaces. *(submitted)*.

Dipartimento di Matematica Pura ed Applicata,
Universita degli Studi di Padova,
via Belzoni 7, I-35131 Padova Italy,
(e-mail: preciso@math.unipd.it).

Department of Mathematics and Mechanics,
Belarusian State University,
F. Skaryny av. 4, BY-220050 Minsk, Belarus,
(email: rogosin@mmf.bsu.unibel.by).